On sharp embeddings of Besov and Triebel-Lizorkin spaces in the subcritical case

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Abstract

We discuss the growth envelopes of Besov and Triebel-Lizorkin spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ for $s = \sigma_p = n\max(\frac{1}{p} - 1, 0)$. These results may be also reformulated as optimal embeddings into the scale of Lorentz spaces $L_{p,q}(\mathbb{R}^n)$. We close several open problems formulated by D. D. Haroske in [4].

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1 Introduction and main results

We denote by $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ the Fourier-analytic Besov and Triebel-Lizorkin spaces (see Definition 2.4 for details). The embeddings of these function spaces (and other spaces of smooth functions) play an important role in functional analysis. If $s > \frac{n}{p}$, then these spaces are continuously embedded into $C(\mathbb{R}^n)$, the space of all complex-valued bounded and uniformly continuous functions on $\mathbb{R}^n$ normed in the usual way. If $s < \frac{n}{p}$ then these function spaces contain also unbounded functions. This statement holds true also for $s = \frac{n}{p}$ under some additional restrictions on the parameters p and q. We refer to [8, Theorem 3.3.1] for a complete overview.

To describe the singularities of these unbounded elements, we use the technique of the non-increasing rearrangement.

Definition 1.1. Let μ be the Lebesgue measure in $\mathbb{R}^n$. If h is a measurable function on $\mathbb{R}^n$, we define the non-increasing rearrangement of h through

$$h^*(t) = \sup\{\lambda > 0 : \mu\{x \in \mathbb{R}^n : |h(x)| > \lambda\} > t\}, \qquad t \in (0,\infty). \tag{1.1}$$

To be able to apply this procedure to elements of $A^s_{p,q}(\mathbb{R}^n)$ (with A standing for B or F), we have to know whether $A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n)$, the space of all measurable, locally-integrable functions on $\mathbb{R}^n$. A complete treatment of this question may be found in [8, Theorem 3.3.2]:

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n) \Leftrightarrow \begin{cases} \text{either} & s > \sigma_p := n\max(\frac{1}{p}-1,0), \\ \text{or} & s = \sigma_p, 1 < p \le \infty, 0 < q \le \min(p,2), \\ \text{or} & s = \sigma_p, 0 < p \le 1, 0 < q \le 1 \end{cases} \tag{1.2}$$

and

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n) \Leftrightarrow \begin{cases} \text{either} & s > \sigma_p, \\ \text{or} & s = \sigma_p, 1 \le p \le \infty, 0 < q \le 2, \\ \text{or} & s = \sigma_p, 0 < p < 1, 0 < q \le \infty. \end{cases} \tag{1.3}$$

Let us assume, that a function space X is embedded into $L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The *growth envelope function* of X was defined by D. D. Haroske (see [3], [4] and references given there) by

$$\mathcal{E}_G^X(t) := \sup_{||f|X|| \le 1} f^*(t), \quad 0 < t < 1.$$

If $\mathcal{E}_G^X(t) \approx t^{-\alpha}$ for $0 < t < 1$ and some $\alpha > 0$, then we define the *growth envelope index* u_X as the infimum of all numbers v, $0 < v \le \infty$, such that

$$\left(\int_0^\epsilon \left[\frac{f^*(t)}{\mathcal{E}_G^X(t)}\right]^v \frac{dt}{t}\right)^{1/v} \le c\,||f|X|| \tag{1.4}$$

holds for some $\epsilon > 0, c > 0$ and all $f \in X$.

The pair $\mathfrak{E}_G(X) = (\mathcal{E}_G^X, u_X)$ is called *growth envelope* for the function space X.

In the case $\sigma_p < s < \frac{n}{p}$, the growth envelopes of $A^s_{p,q}(\mathbb{R}^n)$ are known, cf. [4, Theorem 7.1]. If $s = \frac{n}{p}$ and (1.2) or (1.3) is fulfilled in the B or F case, respectively, then the known information is not complete, cf. [4, Prop. 7.10, 7.12]:

Theorem 1.2. *(i) Let* $1 < p < \infty$ *and* $0 < q \le \min(p,2)$. *Then*

$$\mathfrak{E}_G(B^0_{p,q}) = (t^{-\frac{1}{p}}, u) \qquad \textit{with} \quad q \le u \le p.$$

(ii) Let $1 \le p < \infty$ and $0 < q \le 2$. Then

$$\mathfrak{E}_G(F^0_{p,q}) = (t^{-\frac{1}{p}}, p).$$

(iii) Let $0 < p \le 1, 0 < q \le 1$. Then

$$\mathfrak{E}_G(B^{\sigma_p}_{p,q}) = (t^{-1}, u) \qquad \textit{with} \quad q \le u \le 1.$$

(iv) Let $0 < p < 1$ and $0 < q \le \infty$. Then

$$\mathfrak{E}_G(F^{\sigma_p}_{p,q}) = (t^{-1}, u) \qquad \textit{with} \quad p \le u \le 1.$$

We fill all the above mentioned gaps.

Theorem 1.3. *(i) Let $1 \le p < \infty$ and $0 < q \le \min(p, 2)$. Then*

$$\mathfrak{E}_G(B^0_{p,q}) = (t^{-\frac{1}{p}}, p).$$

(ii) Let $0 < p < 1, 0 < q \le 1$. Then

$$\mathfrak{E}_G(B^{\sigma_p}_{p,q}) = (t^{-1}, q).$$

(iii) Let $0 < p < 1$ and $0 < q \le \infty$. Then

$$\mathfrak{E}_G(F^{\sigma_p}_{p,q}) = (t^{-1}, p).$$

We also reformulate these results as optimal local embeddings into the scale of Lorentz spaces (cf. Definition 2.1):

Theorem 1.4. *(i) Let $1 \le p < \infty$ and $0 < q \le \min(p, 2)$. Then*

$$B^0_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n).$$

(ii) Let $0 < p < 1, 0 < q \le 1$ and $s = \sigma_p$. Then

$$B^{\sigma_p}_{p,q}(\mathbb{R}^d) \hookrightarrow L_{1,q}(\mathbb{R}^n). \tag{1.5}$$

(iii) Let $0 < p < 1$ and $0 < q \le \infty$. Then

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n)$$

and all these embeddings are optimal with respect to the second fine parameter of the scale of the Lorentz spaces.

Remark 1.5. Let us also observe, that (1.5) improves [8, Theorem 3.2.1] and [7, Theorem 2.2.3], where the embedding $B^{n(\frac{1}{p}-1)}_{p,q}(\mathbb{R}^n) \hookrightarrow L_1(\mathbb{R}^n)$ is proved for all $0 < p < 1$ and $0 < q \le 1$.

2 Preliminaries, notation and definitions

We use standard notation: $\mathbb{N}$ denotes the collection of all natural numbers, $\mathbb{R}^n$ is the Euclidean n-dimensional space, where $n \in \mathbb{N}$, and $\mathbb{C}$ stands for the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$ and let $S'(\mathbb{R}^n)$ be its dual - the space of all tempered distributions.

Definition 2.1. (i) Let $0 < p \le \infty$. We denote by $L_p(\mathbb{R}^n)$ the Lebesgue spaces endowed with the quasi-norm

$$||f|L_p(\mathbb{R}^n)|| = \begin{cases} \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}, & 0 < p < \infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|, & p = \infty. \end{cases}$$

(ii) Let $0 < p, q \le \infty$. Then the Lorentz space $L_{p,q}(\mathbb{R}^n)$ consists of all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ such that the quantity

$$||f|L_{p,q}(\mathbb{R}^n)|| = \begin{cases} \left(\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t}\right)^{1/q}, & 0 < q < \infty, \\ \sup_{0<t<\infty} t^{\frac{1}{p}} f^*(t), & q = \infty \end{cases}$$

is finite

Remark 2.2. These definitions are well-known, we refer to [1, Ch 4.4] for details and further references. We shall need only very few properties of these spaces. Obviously, $L_{p,p} = L_p$. If $0 < q_1 \le q_2 \le \infty$, then $L_{p,q_1}(\mathbb{R}^n) \hookrightarrow L_{p,q_2}(\mathbb{R}^n)$ - so the Lorentz spaces are monotonically ordered in q. We shall make us of the following lemma:

Lemma 2.3. *Let $0 < q < 1$. Then the $||\cdot|L_{1,q}(\mathbb{R}^n)||$ is the q-norm, it means*

$$||f_1 + f_2|L_{1,q}(\mathbb{R}^n)||^q \le ||f_1|L_{1,q}(\mathbb{R}^n)||^q + ||f_2|L_{1,q}(\mathbb{R}^n)||^q$$

holds for all $f_1, f_2 \in L_{1,q}(\mathbb{R}^n)$.

For $f \in S'(\mathbb{R}^n)$ we denote by $\widehat{f} = Ff$ its Fourier transform and by $f^\vee$ or $F^{-1}f$ its inverse Fourier transform.

We give a Fourier-analytic definition of Besov and Triebel-Lizorkin spaces, which relies on the so-called *dyadic resolution of unity*. Let $\varphi \in S(\mathbb{R}^n)$ with

$$\varphi(x) = 1 \quad \text{if} \quad |x| \le 1 \quad \text{and} \quad \varphi(x) = 0 \quad \text{if} \quad |x| \ge \frac{3}{2}. \tag{2.1}$$

We put $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}^n$. This leads to the identity

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \qquad x \in \mathbb{R}^n.$$

Definition 2.4. (i) Let $s \in \mathbb{R}, 0 < p, q \le \infty$. Then $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|B^s_{pq}(\mathbb{R}^n)|| = \left(\sum_{j=0}^\infty 2^{jsq} ||(\varphi_j \widehat{f})^\vee |L_p(\mathbb{R}^n)||^q\right)^{1/q} < \infty \tag{2.2}$$

(with the usual modification for $q = \infty$).

(ii) Let $s \in \mathbb{R}, 0 < p < \infty, 0 < q \le \infty$. Then $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in S'(\mathbb{R}^n)$ such that

$$||f|F^s_{pq}(\mathbb{R}^n)|| = \left|\left|\left(\sum_{j=0}^\infty 2^{jsq} |(\varphi_j \widehat{f})^\vee(\cdot)|^q\right)^{1/q} |L_p(\mathbb{R}^n)\right|\right| < \infty \tag{2.3}$$

(with the usual modification for $q = \infty$).

Remark 2.5. These spaces have a long history. In this context we recommend [6], [9], [10] and [12] as standard references. We point out that the spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are independent of the choice of φ in the sense of equivalent (quasi-)norms. Special cases of these two scales include Lebesgue spaces, Sobolev spaces, Hölder-Zygmund spaces and many other important function spaces.

We introduce the sequence spaces associated with the Besov and Triebel-Lizorkin spaces. Let $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. Then $Q_{j\,m}$ denotes the closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes, centred at $2^{-j}m$, and with side length 2^{-j}. By $\chi_{j\,m} = \chi_{Q_{j\,m}}$ we denote the characteristic function of $Q_{j\,m}$. If

$$\lambda = \{\lambda_{j\,m} : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\},$$

$-\infty < s < \infty$ and $0 < p, q \le \infty$, we set

$$||\lambda|b^s_{pq}|| = \Big(\sum_{j=0}^{\infty} 2^{j(s-\frac{n}{p})q}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{j\,m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \tag{2.4}$$

appropriately modified if $p = \infty$ and/or $q = \infty$. If $p < \infty$, we define also

$$||\lambda|f^s_{pq}|| = \Bigg|\Bigg|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} |2^{js}\lambda_{j\,m}\chi_{j\,m}(\cdot)|^q\Big)^{1/q}|L_p(\mathbb{R}^n)\Bigg|\Bigg|. \tag{2.5}$$

The connection between the function spaces $B^s_{pq}(\mathbb{R}^n)$, $F^s_{pq}(\mathbb{R}^n)$ and the sequence spaces b^s_{pq}, f^s_{pq} may be given by various decomposition techniques, we refer to [12, Chapters 2 and 3] for details and further references.

3 Proofs of the main results

3.1 Proof of Theorem 1.3 (i)

In view of Theorem 1.2, it is enough to prove, that for $1 \le p < \infty$ and $0 < q \le \min(p, 2)$ the index u associated to $B^0_{p,q}(\mathbb{R}^n)$ is greater or equal to p.
We assume in contrary that (1.4) is fulfilled for some $0 < v < p$, $\epsilon > 0$, $c > 0$ and all $f \in B^0_{p,q}(\mathbb{R}^n)$.
Let ψ be a non-vanishing C^∞ function in $\mathbb{R}^n$ supported in $[0,1]^n$ with $\int_{\mathbb{R}^n}\psi(x)dx = 0$.
Let $J \in \mathbb{N}$ be such that $2^{-Jn} < \epsilon$ and consider the function

$$f_j = \sum_{m=1}^{2^{(j-J)n}} \lambda_{jm}\psi(2^j(x-(m,0,\dots,0))), \quad j > J, \tag{3.1}$$

where

$$\lambda_{jm} = \frac{1}{m^{\frac{1}{p}}\log^{\frac{1}{v}}(m+1)}.$$

Then (3.1) represents an atomic decomposition of f in the space $B^0_{p,q}(\mathbb{R}^n)$ according to [12] and we obtain (recall that $v < p$)

$$\begin{aligned}||f_j|B^0_{p,q}(\mathbb{R}^n)|| &\lesssim 2^{-j\frac{n}{p}}\left(\sum_{m=1}^{2^{(j-J)n}} \lambda^p_{j,m}\right)^{1/p} \le 2^{-j\frac{n}{p}}\left(\sum_{m=1}^{\infty} m^{-1}(\log(m+1))^{-\frac{p}{v}}\right)^{1/p}\\ &\lesssim 2^{-j\frac{n}{p}}.\end{aligned} \tag{3.2}$$

On the other hand,

$$\left(\int_0^\epsilon \left[f_j^*(t)t^{\frac{1}{p}}\right]^v \frac{dt}{t}\right)^{1/v} \geq \left(\int_0^{2^{-Jn}} f_j^*(t)^v t^{v/p-1}dt\right)^{1/v} \gtrsim \left(\sum_{m=1}^{2^{(j-J)n}} \lambda_{j,m}^v \int_{c\,2^{-jn}(m-1)}^{c\,2^{-jn}m} t^{v/p-1}dt\right)^{1/v}$$

$$\gtrsim \left(\sum_{m=1}^{2^{(j-J)n}} \lambda_{j,m}^v 2^{-jnv/p} m^{v/p-1}\right)^{1/v} = 2^{-j\frac{n}{p}} \left(\sum_{m=1}^{2^{(j-J)n}} \frac{1}{m\log(m+1)}\right)^{1/v}.$$

As the last series is divergent, this is in a contradiction with (3.2) and (1.4) cannot hold for all $f_j, j > J$.

3.2 Proof of Theorem 1.3 (ii)

Let $0 < p < 1$, $0 < q \leq 1$ and $s = \sigma_p = n\left(\frac{1}{p} - 1\right)$. We show that

$$B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n) \hookrightarrow L_{1,q}(\mathbb{R}^n),$$

or, equivalently,

$$\left(\int_0^\infty [tf^*(t)]^q \frac{dt}{t}\right)^{1/q} \leq c\,||f|B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)||, \qquad f \in B^{\frac{n}{p}-n}_{pq}(\mathbb{R}^n).$$

Let

$$f = \sum_{j=0}^\infty f_j = \sum_{j=0}^\infty \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}$$

be the optimal atomic decomposition of an $f \in B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)$, again in the sense of [12]. Then

$$||f|B^{\frac{n}{p}-n}_{p,q}(\mathbb{R}^n)|| \approx \left(\sum_{j=0}^\infty 2^{-jqn}\left(\sum_{m\in\mathbb{Z}^n} |\lambda_{jm}|^p\right)^{q/p}\right)^{1/q} \tag{3.3}$$

and by Lemma 2.3

$$||f|L_{1,q}(\mathbb{R}^n)|| = ||\sum_{j=0}^\infty f_j|L_{1,q}(\mathbb{R}^n)|| \leq \left(\sum_{j=0}^\infty ||f_j|L_{1,q}(\mathbb{R}^n)||^q\right)^{1/q}. \tag{3.4}$$

We shall need only one property of the atoms $a_{j,m}$, namely that their support is contained in the cube $\tilde{Q}_{j,m}$ - a cube centred at the point $2^{-j}m$ with sides parallel to the coordinate axes and side length $\alpha 2^{-j}$, where $\alpha > 1$ is fixed and independent of f. We denote by $\tilde{\chi}_{j,m}(x)$ the characteristic functions of $\tilde{Q}_{j,m}$ and by $\chi_{j,l}$ the characteristic function of the interval $(l2^{-jn}, (l+1)2^{-jn})$. Hence

$$f_j(x) \leq c \sum_{m\in\mathbb{Z}^n} |\lambda_{j,m}|\tilde{\chi}_{j,m}(x), \quad x \in \mathbb{R}^n$$

and

$$||f_j|L_{1,q}(\mathbb{R}^n)|| \lesssim \left(\int_0^\infty \sum_{l=0}^\infty [(\lambda_j)_l^*\chi_{j,l}(t)]^q\, t^{q-1}dt\right)^{1/q} \leq \left(\sum_{l=0}^\infty [(\lambda_j)_l^*]^q \int_{2^{-jn}l}^{2^{-jn}(l+1)} t^{q-1}dt\right)$$

$$\leq c\,2^{-jn}\left(\sum_{l=0}^\infty [(\lambda_j)_l^*]^q\,(l+1)^{q-1}\right)^{1/q} \leq c\,2^{-jn}||\lambda_j|\ell_p||. \tag{3.5}$$

The last inequality follows by $(l+1)^{q-1} \le 1$ and $\ell_p \hookrightarrow \ell_q$ if $p \le q$. If $p > q$, the same follows by Hölder's inequality with respect to indices $\alpha = \frac{p}{q}$ and $\alpha' = \frac{p}{p-q}$:

$$\left(\sum_{l=0}^{\infty} [(\lambda_j)_l^*]^q \, (l+1)^{q-1}\right)^{1/q} \le \left(\sum_{l=0}^{\infty} [(\lambda_j)_l^*]^{q\cdot\frac{p}{q}}\right)^{\frac{1}{q}\cdot\frac{q}{p}} \cdot \left(\sum_{l=0}^{\infty} (l+1)^{(q-1)\cdot\frac{p}{p-q}}\right)^{\frac{1}{q}\cdot\frac{p-q}{p}} \le c\,||\lambda_j|\ell_p||.$$

The proof now follows by (3.3), (3.4) and (3.5).

$$||f|L_{1,q}(\mathbb{R}^n)|| \le \left(\sum_{j=0}^{\infty} ||f_j|L_{1,q}(\mathbb{R}^n)||^q\right)^{1/q} \le c\left(\sum_{j=0}^{\infty} 2^{-jnq}||\lambda_j|\ell_p||^q\right)^{1/q} \le c\,||f|B^{\sigma_p}_{p,q}(\mathbb{R}^n)||.$$

3.3 Proof of Theorem 1.3 (iii)

Let $0 < p < 1$ and $0 < q \le \infty$. By the Jawerth embedding (cf. [5] or [13]) and Theorem 1.3 (ii) we get for any $0 < p < \tilde{p} < 1$

$$F^{\sigma_p}_{p,q}(\mathbb{R}^n) \hookrightarrow B^{\sigma_{\tilde{p}}}_{\tilde{p},p}(\mathbb{R}^n) \hookrightarrow L_{1,p}(\mathbb{R}^n).$$

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