A REMARK ON BETTER λ-INEQUALITY

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Abstract

We generalize the inequality of R. J. Bagby and D. S. Kurtz [BK] to a wider class of potentials defined in terms of Young's functions. We make use of a certain submultiplicativity condition. We show that this condition cannot be omited.

Key words: Riesz potentials, Better λ-inequality, Nonincreasing rearrangement, Young's functions

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1. Introduction

The classical Riesz potentials are defined for every real number $0 < \gamma < n$ as a convolution operators $(I_\gamma f)(x) = (\tilde{I}_\gamma * f)(x)$, where $\tilde{I}_\gamma(x) = |x|^{\gamma-n}$. This definition coincides with the usual one up to some multiplicative constant c_γ which is not interesting for our purpose. Burkholder and Gundy invented in [BG] the technique involving distribution function later known as *good λ-inequality.* This inequality dealt with level sets of singular integral operators and of maximal operator. Later, Bagby and Kurtz discovered in [BK] that the reformulation of good λ-inequality in terms of non-increasing rearrangement contains more information.

We generalize their approach in the following way. For every Young's function Φ satisfying the Δ_2-condition we define the Riesz potential

$$(I_\Phi f)(x) = \int_{\mathbf{R}^n} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y,$$

where $\tilde{\Phi}$ is Young's function conjugated to Φ and $\tilde{\Phi}^{-1}$ is its inverse. Instead of the classical Hardy-Littlewood maximal operator we work with a generalized maximal operator

$$(M_\varphi f)(x) = \sup_{Q \ni x} \frac{1}{\varphi(|Q|)} \int_Q |f(y)|\mathrm{d}y,$$

where φ is a given nonnegative function on $(0,\infty)$ and the supremum is taken over all cubes Q containing x with sides parallel to the coordinate axes such that $\varphi(|Q|) > 0$. For every measurable set $\Omega \subset \mathbf{R}^n$ we denote by $|\Omega|$ its Lebesgue measure.

We prove that under some restrictive condition on function Φ one can obtain an inequality combining the nonincreasing rearrangement of $I_\Phi f$ and $M_{\tilde{\Phi}^{-1}} f$. We also show that this restrictive condition cannot be left out.

2. Better λ—inequality

Before we state our main result, we give some definitions and recall some very well known results about Young's functions and non-increasing rearrangements.

Lebesgue measure will be denoted by μ or simply be an absolute value. Let Ω be a subset of $\mathbf{R}^n$, $n \geq 1$. We denote by $\mathfrak{M}$ the collection of all extended scalar-valued Lebesgue measurable functions on Ω and by $\mathfrak{M}_0$ the class of functions in $\mathfrak{M}$ that are finite μ-a.e. Further let $\mathfrak{M}^+$ be the cone of nonnegative functions from $\mathfrak{M}$ and $\mathfrak{M}_0^+$ the class of nonnegative functions from $\mathfrak{M}_0$. We shall also write $\mathfrak{M}(\Omega), \mathfrak{M}^+(\Omega)$ and so on when we want to emphasize the underlying space Ω.

The letter c denotes a general constant which doesn't depend on the parameters involved. It may change from one occurrence to another.

Definition 2.1. 1. Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing and right-continuous function with $\phi(0) = 0$ and $\phi(\infty) = \lim_{t\to\infty} \phi(t) = \infty$. Then the function Φ defined by

$$\Phi(t) = \int_0^t \phi(s)\mathrm{d}s, \quad t \geq 0$$

is said to be a *Young's function.*

2. A Young's function is said to satisfy the Δ_2—*condition* if there is $c > 0$ such that

$$\Phi(2t) \leq c\ \Phi(t), \quad t \geq 0.$$

3. A Young's function is said to satisfy the ∇_2—*condition* if there is $l > 1$ such that

$$\Phi(t) \leq \frac{1}{2l}\Phi(lt), \quad t \geq 0.$$

4. Let Φ be a Young's function, represented as the indefinite integral of ϕ. Let

$$\psi(s) = \sup\{u : \phi(u) \leq s\}, \quad s \geq 0.$$

Then the function

$$\tilde{\Phi}(t) = \int_0^t \psi(s)\mathrm{d}s, \quad t \geq 0,$$

is called the *complementary Young's function* of Φ.

The following theorem puts these three notions together. For the proof see [KR].

Theorem 2.2. *Let Φ be a Young's function and $\tilde{\Phi}$ be its complementary Young's function. Then Φ satisfies the Δ_2—condition if and only if $\tilde{\Phi}$ satisfies the ∇_2—condition.*

We shall need following lemma.

Lemma 2.3. *Let Φ be a Young's function satisfying the Δ_2—condition. Then there is a constant $c > 0$ such that*

$$\int_0^t \tilde{\Phi}^{-1}\left(\frac{1}{u}\right)\mathrm{d}u \leq c\,t\tilde{\Phi}^{-1}\left(\frac{1}{t}\right), \quad 0 < t < \infty$$

Proof. If Φ satisfies the Δ_2—condition, then $\tilde{\Phi}$ satisfies the ∇_2—condition. It means that there is a real number $k > 1$ such that $\tilde{\Phi}(t) \leq \frac{1}{2k}\tilde{\Phi}(kt)$ for every $t > 0$. When we pass to inverses we get $\tilde{\Phi}^{-1}\left(\frac{1}{u}\right) \leq \frac{l}{2}\tilde{\Phi}^{-1}\left(\frac{1}{lu}\right)$, where $l = 2k > 2$ and $u > 0$. Now setting $h(s) = \tilde{\Phi}^{-1}\left(\frac{1}{s}\right)$ and $H(u) = \int_0^u h(s)\mathrm{d}s$ we get $2h(s) \leq lh(ls)$ and integrating this inequality from 0 to t we obtain $2H(t) \leq H(lt)$. To show that $H(t)$ is finite for all $t > 0$, write

$$H(t) = \int_0^t h(s)\mathrm{d}s = \sum_{k=0}^{\infty}\int_{t/l^{k+1}}^{t/l^k} h(s)\mathrm{d}s \leq \sum_{k=0}^{\infty}\int_{t/l^{k+1}}^{t/l^k} \frac{l^k}{2^k}h(l^k s)\mathrm{d}s =$$

$$= \sum_{k=0}^{\infty}\frac{1}{2^k}\int_{t/l}^{t} h(u)\mathrm{d}u < \infty.$$

Because h is a decreasing function, we can calculate

$$lth(t) \geq \int_t^{lt} h(s)\mathrm{d}s = H(lt) - H(t) \geq 2H(t) - H(t) = H(t),$$

which can be rewritten as

$$lt\tilde{\Phi}^{-1}\left(\frac{1}{t}\right) \geq \int_0^t \tilde{\Phi}^{-1}\left(\frac{1}{u}\right) \mathrm{d}u.$$

□

Definition 2.4. The *distribution function* μ_f of a function f in $\mathfrak{M}_0(\Omega)$ is given by

$$\mu_f(\lambda) = \mu(\{x \in \Omega : |f(x)| > \lambda\}), \quad \lambda \geq 0.$$

For every $f \in \mathfrak{M}_0(\Omega)$ we define its *nonincreasing rearrangement* f^* by

$$f^*(t) = \inf\{\lambda : \mu_f(\lambda) \leq t\}, \quad 0 \leq t < \infty$$

and its *maximal function* f^{**} by

$$f^{**}(t) = t^{-1}\int_0^t f^*(u)\mathrm{d}u, \quad 0 < t < \infty.$$

Assume now that Young's function Φ satisfies the Δ_2—condition. Using the classical O'Neil inequality (see [O]) and lemma 2.3 we obtain

$$(I_\Phi f)^*(t) \leq c\left\{\tilde{\Phi}^{-1}\left(\frac{1}{t}\right)\int_0^t f^*(u)\mathrm{d}u + \int_t^\infty f^*(u)\tilde{\Phi}^{-1}\left(\frac{1}{u}\right)\mathrm{d}u\right\}, \tag{1}$$

We shall derive a better λ-inequality connecting the operators I_Φ and $M_{\tilde{\Phi}^{-1}}$.

Theorem 2.5. *Let us suppose that a Young's function Φ satisfies the Δ_2—condition. Let us further suppose that there is a constant $c_1 > 0$ such that*

$$\tilde{\Phi}^{-1}(s)\tilde{\Phi}^{-1}(1/s) < c_1, \quad s > 0. \tag{2}$$

Then there is a constant $c_2 > 0$, such that for every function f and every positive number t

$$(I_\Phi f)^*(t) \leq (I_\Phi|f|)^*(t) \leq c_2\,(M_{\tilde{\Phi}^{-1}}f)^*(t/2) + (I_\Phi|f|)^*(2t) \tag{3}$$

Proof. We may assume that given function f is nonnegative.

First we shall estimate the size of the level set $G = \{x \in \mathbf{R}^n : (I_\Phi g)(x) > \lambda\}$ for function $g \in L^1(\mathbf{R}^n)$. According to (1), $|G| < \infty$. Hence we can find a real number $R \geq 0$ such that $|G| = |B(0,R)|$. We can write

$$\begin{aligned}
\lambda|G| &= \int_G \lambda \leq \int_G (I_\Phi g)(x)\mathrm{d}x = \int_G\int_{\mathbf{R}^n} g(y)\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right)\mathrm{d}y\mathrm{d}x = \\
&= \int_{\mathbf{R}^n}\int_G \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right)\mathrm{d}x g(y)\mathrm{d}y \leq \\
&||g||_1\int_{B(0,R)}\tilde{\Phi}^{-1}\left(\frac{1}{|x|^n}\right)\mathrm{d}x = ||g||_1\alpha_n\int_0^{|G|/\alpha_n}\tilde{\Phi}^{-1}(1/s)\mathrm{d}s.
\end{aligned}$$

Dividing this inequality by $|G|$ and using the lemma 2.3 we obtain

$$\lambda \leq ||g||_1\frac{\alpha_n}{|G|}\int_0^{|G|/\alpha_n}\tilde{\Phi}^{-1}(1/s)\mathrm{d}s \leq \tilde{c}\,||g||_1\tilde{\Phi}^{-1}\left(\frac{1}{|G|}\right).$$

This can be rewritten as

$$|G| \leq \frac{1}{\tilde{\Phi}\left(\frac{\lambda}{\tilde{c}||g||_1}\right)}, \tag{4}$$

where $\tilde{c}$ is independent of g and λ.

We can now pass to the proof of our theorem which is mainly based on [BK]. For a given function $f \geq 0$ and a real number $t > 0$ we shall denote by E the set $\{x \in \mathbf{R}^n : (I_\Phi f)(x) > (I_\Phi f)^*(2t)\}$. Then $|E| \leq 2t$ and we can find an open set Ω,

$|\Omega| < 3t, E \subset \Omega$. Now using Whitney covering theorem (see [S]) we can find cubes Q_k with disjoint interiors, such that $\Omega = \cup_{k=1}^{\infty} Q_k$ and $\operatorname{diam} Q_k \le \operatorname{dist}(Q_k, \mathbf{R}^n \setminus \Omega) \le 4 \operatorname{diam} Q_k$.

We want to show that there is a constant $C > 0$ such that for every f, t and for every corresponding cube Q_k

$$|\{x \in Q_k : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| \le \frac{1}{6}|Q_k|. \tag{5}$$

Then we would have $|\{x \in \mathbf{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| \le 1/6 \sum |Q_k| \le t/2$ and thus

$$\begin{aligned} &|\{x \in \mathbf{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)^*(t/2) + (I_\Phi f)^*(2t)\}| \le \\ &\quad \le |\{x \in \mathbf{R}^n : I_\Phi f(x) > C(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t)\}| + \\ &\quad\quad + |\{x \in \mathbf{R}^n : (M_{\tilde{\Phi}^{-1}} f)(x) > (M_{\tilde{\Phi}^{-1}} f)^*(t/2)\}| \le t/2 + t/2 = t, \end{aligned}$$

which finishes the proof.

To prove (5) fix k and choose $x_k \in (\mathbf{R}^n \setminus \Omega)$ so that $\operatorname{dist}(x_k, Q_k) \le 4 \operatorname{diam}(Q_k)$. Let Q be a cube with center at x_k having diameter $20 \operatorname{diam}(Q_k)$. Split $f = g + h = f\chi_Q + f\chi_{\mathbf{R}^n \setminus Q}$. We may assume that $g \in L^1(\mathbf{R}^n)$, otherwise the right-hand side of (3) would be infinite.

We shall prove that for C_1 and C_2 large enough

$$|\{x \in Q_k : (I_\Phi g)(x) > C_1(M_{\tilde{\Phi}^{-1}} f)(x)\}| \le 1/6|Q_k|, \tag{6}$$

and, for every $x \in Q_k$,

$$I_\Phi h(x) \le C_2(M_{\tilde{\Phi}^{-1}} f)(x) + I_\Phi f(x_k) \le C_2(M_{\tilde{\Phi}^{-1}} f)(x) + (I_\Phi f)^*(2t), \tag{7}$$

which together gives (5).

For the first inequality, notice that for $x \in Q_k$

$$(M_{\tilde{\Phi}^{-1}} f)(x) \ge \frac{1}{\tilde{\Phi}^{-1}(|Q|)} \int_Q g = \frac{||g||_1}{\tilde{\Phi}^{-1}(|Q|)}.$$

Using (4) now gives

$$\begin{aligned} |\{x \in Q_k :&(I_\Phi g)(x) > C_1(M_{\tilde{\Phi}^{-1}} f)(x)\}| \le \\ &\left|\left\{x \in Q_k : (I_\Phi g)(x) > \frac{C_1 ||g||_1}{\tilde{\Phi}^{-1}(|Q|)}\right\}\right| \le \frac{1}{\tilde{\Phi}\left(\frac{C_1}{\tilde{c}\tilde{\Phi}^{-1}(|Q|)}\right)}, \end{aligned}$$

where $\tilde{c}$ is the constant from (4). The last expression is less then $|Q_k|/6$ for C_1 big enough (here we use (2) again).

In the proof of the second inequality we shall use two observations. The first is that

$$\left|\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) - \tilde{\Phi}^{-1}\left(\frac{1}{|x_k-y|^n}\right)\right| \le c\,\frac{|x_k - x|}{|x-y|}\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \tag{8}$$

with c independent of k, $y \in (\mathbf{R}^n \setminus Q)$ and $x \in Q_k$.

The second is that for any $\delta > 0$ and any $x \in \mathbf{R}^n$

$$\int_{y:|x-y|>\delta} \delta \frac{f(y)}{|x-y|}\tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) \mathrm{d}y \le c\, M_{\tilde{\Phi}^{-1}} f(x). \tag{9}$$

The proof of (7) now follows easily. For every $x \in Q_k$ we get

$$\begin{aligned}
&I_\Phi h(x) - I_\Phi f(x_k) \leq I_\Phi h(x) - I_\Phi h(x_k) \leq \\
&\int_{\mathbf{R}^n \setminus Q} \left| \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) - \tilde{\Phi}^{-1}\left(\frac{1}{|x_k-y|^n}\right) \right| f(y)\mathrm{d}y \leq \\
&c|x_k - x| \int_{\mathbf{R}^n \setminus Q} \frac{1}{|x-y|} \tilde{\Phi}^{-1}\left(\frac{1}{|x-y|^n}\right) f(y)\mathrm{d}y \leq \\
&cM_{\tilde{\Phi}^{-1}} f(x).
\end{aligned}$$

It remains to prove (8) and (9). Proof of (9) is a combination of definition of $M_{\tilde{\Phi}^{-1}}$ and (2).

To prove (8) let us write $\tilde{\Phi}(t) = \int_0^t \tilde{\varphi}(u)\mathrm{d}u$ and $A(t) = \tilde{\Phi}^{-1}(t^{-n})$ for $t > 0$. Then

$$\frac{1}{s}\int_0^s \tilde{\varphi}(u)\mathrm{d}u \leq \tilde{\varphi}(s), \quad s > 0$$

or, equivalently, $\tilde{\Phi}(s) \leq s\tilde{\Phi}'(s)$ for $s > 0$. Now we set $s = A(t)$ and obtain

$$-tA'(t) = \frac{nt^{-n}}{\tilde{\Phi}'(A(t))} \leq cA(t).$$

Finally the left hand side of (8) can be estimated by

$$|A(|x-y|) - A(|x_k - y|)| \leq c\left| \int_{|x-y|}^{|x_k-y|} \frac{A(t)}{t}\mathrm{d}t \right| \leq c\frac{|x_k - x|}{|x-y|}A(|x-y|).$$

$\square$

In the following example we will show that the assumption (2) cannot be omitted.

Theorem 2.6. *There is a Young's function Φ satisfying the Δ_2—condition for which*

$$\sup_{f,t>0} \frac{(I_\Phi f)^*(t) - (I_\Phi f)^*(2t)}{(M_{\tilde{\Phi}^{-1}} f)^*(t/2)} = \infty$$

Proof. Set

$$\tilde{\Phi}(u) = \begin{cases} u^3 & \text{if } 0 < u < 1 \\ \frac{3}{2}u^2 - \frac{1}{2} & \text{if } 1 < u < \infty \end{cases}, \qquad \tilde{\varphi}(u) = \begin{cases} 3u^2 & \text{if } 0 < u < 1 \\ 3u & \text{if } 1 < u < \infty \end{cases}.$$

Then

$$\Phi(u) = \begin{cases} \frac{2}{3\sqrt{3}}u^{3/2} & \text{if } 0 < u < 3 \\ \frac{u^2}{6} + \frac{1}{2} & \text{if } 3 < u < \infty \end{cases}, \qquad \varphi(u) = \begin{cases} \sqrt{\frac{u}{3}} & \text{if } 0 < u < 3 \\ \frac{u}{3} & \text{if } 3 < u < \infty \end{cases}.$$

Finally $\tilde{\Phi}^{-1}(u) = \sqrt[3]{u}$ for $0 < u < 1$ and $\tilde{\Phi}^{-1}(u) = \sqrt{2/3(u + 1/2)}$ for $u > 1$.

Let $n = 1$. For any integer $m > 0$ set $t_m = 1/m$, $f_m(x) = \chi_{(0,t_m)}(x)$. Then

$$(M_{\tilde{\Phi}^{-1}} f_m)^*(t_m/2) = (M_{\tilde{\Phi}^{-1}} f_m)(0) = \sup_{0<s<1/m} \frac{1}{\tilde{\Phi}^{-1}(s)} \int_0^s 1 = m^{-2/3},$$

$$(I_\Phi f_m)^*(t_m) = (I_\Phi f_m)(0) = \int_0^{1/m} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s = \sqrt{\frac{2}{3}} \int_0^{1/m} \sqrt{\frac{1}{u} + \frac{1}{2}}\mathrm{d}u,$$

$$(I_\Phi f_m)^*(2t_m) = (I_\Phi f_m)\left(\frac{3}{2}t_m\right) = \int_{1/(2m)}^{3/(2m)} \tilde{\Phi}^{-1}(1/s)\mathrm{d}s = \sqrt{\frac{2}{3}} \int_{1/(2m)}^{3/(2m)} \sqrt{\frac{1}{u} + \frac{1}{2}}\mathrm{d}u.$$

We can now estimate

$$\frac{(I_\Phi f_m)^*(t_m) - (I_\Phi f_m)^*(2t_m)}{(M_{\tilde{\Phi}^{-1}} f_m)^*(t_m/2)} \geq$$

$$\sqrt{\frac{2}{3}} m^{2/3} \left\{ \int_0^{1/(2m)} \sqrt{\frac{1}{u}} \mathrm{d}u - \int_{1/m}^{3/(2m)} \sqrt{m + \frac{1}{2}} \mathrm{d}u \right\} =$$

$$\sqrt{\frac{2}{3}} m^{2/3} \left\{ \frac{\sqrt{2}}{\sqrt{m}} - \frac{\sqrt{m+\frac{1}{2}}}{2m} \right\} = \sqrt{\frac{2}{3}} m^{1/6} \left\{ \sqrt{2} - \frac{1}{2}\sqrt{1 + \frac{1}{2m}} \right\}.$$

The last expression tends to infinity as m tends to infinity. □

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