

# AN UPPER BOUND ON THE MINIMAL DISPERSION

MARIO ULLRICH AND JAN VYBÍRAL

ABSTRACT. For  $\varepsilon \in (0, 1/2)$  and a natural number  $d \geq 2$ , let  $N$  be a natural number with

$$N \geq 2^9 \log_2(d) \left( \frac{\log_2(1/\varepsilon)}{\varepsilon} \right)^2.$$

We prove that there is a set of  $N$  points in the unit cube  $[0, 1]^d$ , which intersects all axis-parallel boxes with volume  $\varepsilon$ . That is, the dispersion of this point set is bounded from above by  $\varepsilon$ .

## 1. INTRODUCTION

We are interested in bounds on the volume of the largest axis-parallel box that does not contain any point from a given finite point set  $\mathcal{P} \subset [0, 1]^d$ . Moreover, we would like to find a point set such that this volume is as small as possible. To be precise, we define, for  $d \in \mathbb{N}$  and a point set  $\mathcal{P} \subset [0, 1]^d$ , the *dispersion of  $\mathcal{P}$*  by

$$\text{disp}(\mathcal{P}) := \sup_{B: B \cap \mathcal{P} = \emptyset} |B|,$$

where the supremum is over all axis-parallel boxes  $B = I_1 \times \cdots \times I_d$  with intervals  $I_\ell \subset [0, 1]$ , and  $|B|$  denotes the (Lebesgue) volume of  $B$ . Moreover, for  $n, d \in \mathbb{N}$ , let the  *$n$ th-minimal dispersion* be defined by

$$\text{disp}(n, d) := \inf_{\substack{\mathcal{P} \subset [0, 1]^d: \\ \#\mathcal{P} = n}} \text{disp}(\mathcal{P})$$

and define its inverse function

$$N(\varepsilon, d) := \min \left\{ n : \text{disp}(n, d) \leq \varepsilon \right\}.$$

These quantities were introduced by Rote and Tichy [14] (as a modification of a quantity considered by Hlawka [6]) and attracted quite a lot of attention in the past years in the context of information-based complexity theory, where the explicit dependence of certain geometric quantities on the dimension  $d$  plays a crucial role. Bounds on the dispersion (or any of its variants) translate into bounds on worst-case errors (and

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hence complexity bounds) for several numerical problems. These include optimization in different settings [9, 21], approximation of high-dimensional rank-1 tensors [2, 12] and, very recently, approximation of  $L_p$ -norms and Marcinkiewicz-type discretization [17, 18, 19]. However, it is still not clear so far, if there exists a numerical problem that corresponds to the dispersion in the same way as the *discrepancy* corresponds to numerical integration, see e.g. [3, 4, 5, 10, 11, 13].

Besides this, the dispersion is clearly an interesting geometric quantity on its own. It is easy to define and one might think it is also simple to tackle. But, as the dispersion still resists a precise analysis, this does not seem to be the case. However, there are several upper and lower bounds on the minimal dispersion, most of which were established in the past three years. Here we comment briefly on the state of the art.

First of all, it is quite easy to see that the minimal dispersion is of order  $n^{-1}$  for all  $d$ . The best bounds of this order so far, which show also an explicit dependence on  $d$ , are

$$\frac{\log_2(d)}{4(n + \log_2(d))} \leq \text{disp}(n, d) \leq \frac{C^d}{n}$$

for some constant  $C < \infty$ . The lower bound is due to Aistleitner et al. [1] and the upper bound was obtained by Larcher [8] (see [1, Section 4] for the proof). Concerning the dependence on the dimension  $d$ , we see that the above bounds are far from being tight. However, it was recently proved by Sosnovec [16], that (surprisingly) the logarithmic dependence in the lower bound is sharp. He proved that, for every fixed  $\varepsilon > 0$ ,

$$N(\varepsilon, d) \leq c_\varepsilon \log_2(d).$$

However, in this bound the  $\varepsilon$ -dependence is far off. Further results on the dispersion are polynomial (in  $d$  and  $1/\varepsilon$ ) bounds by Rudolf [15] (see Remark 1) and an explicit construction based on sparse grids by Krieg [7]. Interestingly, a lower bound linear in  $d$  was recently obtained by one of the authors [20] in the periodic setting.

It seems reasonable to conjecture that  $\text{disp}(n, d) \asymp \log(d)/n$ . However, it is not yet clear if this bound can hold for all  $n$  and  $d$ .

In this article we refine the analysis of [16] paying attention to the  $\varepsilon$ -dependence and narrow the existing gap. We prove an upper bound on the inverse of the minimal dispersion that is logarithmic in  $d$  and almost quadratic in  $1/\varepsilon$ .

**Theorem 1.** *Let  $d \geq 2$  be a natural number and let  $\varepsilon \in (0, 1/2)$ . Then there exists a point set  $\mathcal{P} \subset [0, 1]^d$  with  $\text{disp}(\mathcal{P}) \leq \varepsilon$  and*

$$\#\mathcal{P} \leq 2^7 \log_2(d) \frac{(1 + \log_2(\varepsilon^{-1}))^2}{\varepsilon^2}.$$

Clearly, the right hand side is bounded above by the  $N$  given in the abstract. Moreover, Theorem 1 directly implies the following.

**Corollary 2.** *For  $n, d \in \mathbb{N}$  with  $n \geq 2$  and  $d \geq 2$  we have*

$$\text{disp}(n, d) \leq c \log_2(n) \sqrt{\frac{\log_2(d)}{n}}$$

for some absolute constant  $c > 0$ .

## 2. PROOF

We will now prove Theorem 1. For this, we have to prove that there exists a point set  $\mathcal{P}$  with the desired cardinality that has dispersion bounded by  $\varepsilon$ , i.e., every box of volume  $\varepsilon$  contains at least one point from  $\mathcal{P}$ .

For  $0 < \varepsilon < 1/2$ , let  $k \in \mathbb{N}$  with  $2^{-k} \leq \varepsilon < 2^{-k+1}$ , i.e.,  $k = \lceil \log_2(1/\varepsilon) \rceil \geq 2$ , and define

$$M_k = \left\{ \frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k - 1}{2^k} \right\} \subset [0, 1].$$

We consider the random point set  $X = \{x^1, x^2, \dots, x^n\} \subset M_k^d$  with the coordinates  $(x^j)_\ell$  being chosen independently and uniformly from  $M_k$ . We show that for  $n$  growing polynomially in  $1/\varepsilon$  and logarithmically in  $d$ ,  $X$  intersects every cube with sides parallel to the coordinate axis and volume at least  $\varepsilon$  with positive probability. This proves our existence result.

We begin with splitting the set of all boxes of volume at least  $2^{-k}$  into several groups. This is necessary for the following union bound. For  $p = (p_\ell)_{\ell=1}^d \in M_k^d$  and  $s = (s_\ell)_{\ell=1}^d \in \{0, \dots, 2^k - 1\}^d$  let

$$\Omega_k := \{B \subset [0, 1]^d : B \text{ is an axis-parallel box with } |B| > 2^{-k}\}$$

and

$$(1) \quad \Omega_k(p, s) := \left\{ I_1 \times \dots \times I_d \in \Omega_k : \frac{s_\ell}{2^k} < |I_\ell| \leq \frac{s_\ell + 1}{2^k} \right. \\ \left. \text{and } \inf I_\ell \in \left[ p_\ell - \frac{1}{2^k}, p_\ell \right) \text{ for all } \ell = 1, \dots, d \right\}.$$

Clearly,  $\Omega_k(p, s) = \emptyset$  if  $s_\ell = 0$  for some  $\ell = 1, \dots, d$ .

The proof of the main result is based, inter alia, on the following lemma, which will be proved at the end of this section.

**Lemma 3.** *Let  $x$  be uniformly distributed in  $M_k^d$ . Then, for each  $B \in \Omega_k$ ,*

$$\mathbb{P}(x \in B) > 2^{-k-4}.$$

Moreover, for each  $p \in M_k^d$  and  $s \in \{1, \dots, 2^k - 1\}^d$ , we have

$$\mathbb{P}(\exists B \in \Omega_k(p, s) : x \notin B) < \exp(-2^{-k-4}).$$

*Proof of Theorem 1.* Recall that  $X = \{x^1, x^2, \dots, x^n\} \subset M_k^d$  is our random point set. By a simple union bound and Lemma 3, we obtain

$$\begin{aligned} \mathbb{P}(\exists B \in \Omega_k: X \cap B = \emptyset) &\leq \sum_{p,s: \Omega_k(p,s) \neq \emptyset} \mathbb{P}(\exists B \in \Omega_k(p,s): X \cap B = \emptyset) \\ &= \sum_{p,s: \Omega_k(p,s) \neq \emptyset} \mathbb{P}(\exists B \in \Omega_k(p,s): x^1 \notin B)^n \\ &< \#\{(p,s): \Omega_k(p,s) \neq \emptyset\} \exp(-n 2^{-k-4}). \end{aligned}$$

To estimate further, we need to bound from above the number of pairs  $(p,s) \in M_k^d \times \{0, \dots, 2^k - 1\}^d$ , for which  $\Omega_k(p,s)$  is non-empty. We observe, that it is impossible to find an interval  $I_\ell \subset [0, 1]$  with  $|I_\ell| > \frac{s_\ell}{2^k}$  and  $\inf I_\ell \geq p_\ell - \frac{1}{2^k}$  if  $p_\ell - \frac{1}{2^k} + \frac{s_\ell}{2^k} \geq 1$ . Therefore,  $\Omega_k(p,s)$  is empty if  $p_\ell 2^k \geq 2^k + 1 - s_\ell$  for some  $\ell = 1, \dots, d$ .

Furthermore, this implies that if  $s$  is such that  $\Omega_k(p,s) \neq \emptyset$  for some  $p \in M_k^d$ , then there are exactly  $\prod_{\ell=1}^d (2^k - s_\ell)$  choices for  $p$  with  $\Omega_k(p,s) \neq \emptyset$ , i.e.,  $\#\{p: \Omega_k(p,s) \neq \emptyset\} = \prod_{\ell=1}^d (2^k - s_\ell)$ . Denoting  $m_1(s) := \#\{\ell: s_\ell < 2^k - 1\}$ , we see that

$$\#\{p: \Omega_k(p,s) \neq \emptyset\} \leq 2^{km_1(s)}.$$

Now note that, for  $B \in \Omega_k(p,s)$ , we have

$$2^{-k} < |B| \leq \prod_{\ell=1}^d \left( \frac{s_\ell + 1}{2^k} \right) \leq \left( 1 - \frac{1}{2^k} \right)^{m_1(s)}.$$

Therefore,  $\Omega_k(p,s) \neq \emptyset$  implies that

$$(2) \quad m_1(s) < A_k := \ln(2) k 2^k.$$

Indeed, we have

$$2^{-k} < \left( 1 - \frac{1}{2^k} \right)^{m_1(s)} \quad \text{and} \quad 2^{-k} > \left( 1 - \frac{1}{2^k} \right)^{\ln(2) k 2^k}.$$

The latter of these two formulas follows by a monotone convergence of  $(1 - 1/2^k)^{2^k}$  up to  $e^{-1}$ .

The number of  $s \in \{0, \dots, 2^k - 1\}^d$  with  $m_1(s) < A_k$  is bounded by

$$\binom{d}{A_k} 2^{kA_k} < \left( \frac{4d}{k} \right)^{A_k},$$

where we use  $\binom{d}{A_k} \leq (ed/A_k)^{A_k}$  and  $e/\ln(2) < 4$ . We obtain

$$\begin{aligned} \#\{(p,s): \Omega_k(p,s) \neq \emptyset\} &< \left( \frac{4d}{k} \right)^{A_k} 2^{kA_k} \leq \exp\left(k 2^k \left(k + \log_2(4d/k)\right)\right) \\ &\leq \exp\left(k 2^k \log_2(2^{k+1}d)\right) \end{aligned}$$

and

$$\mathbb{P}(\exists B \in \Omega_k: X \cap B = \emptyset) < \exp\left(k 2^k \log_2(2^{k+1}d) - n 2^{-k-4}\right),$$

which is smaller than one if

$$n \geq 2^4 k 2^{2k} \log_2(2^{k+1}d).$$

This ensures the existence of a set  $X$  with  $n$  points, for which  $X \cap B \neq \emptyset$  for all cubes  $B$  with  $|B| > 2^{-k}$ . Therefore,

$$N(2^{-k}, d) \leq 2^4 k 2^{2k} \log_2(2^{k+1}d).$$

Finally, from  $2^{-k} \leq \varepsilon < 2^{-k+1}$ ,  $2^{k-1} < \varepsilon^{-1} \leq 2^k$  and  $k \geq \log_2(1/\varepsilon) > k-1$ , we get that

$$\begin{aligned} N(\varepsilon, d) &\leq 2^6 \frac{(1 + \log_2(\varepsilon^{-1})) \log_2(4d\varepsilon^{-1})}{\varepsilon^2} \\ &\leq 2^7 \log_2(d) \frac{(1 + \log_2(\varepsilon^{-1}))^2}{\varepsilon^2}. \end{aligned}$$

□

For the proof of Theorem 1 it remains to prove Lemma 3. But before that, we state an alternative bound.

*Remark 1.* If we repeat the above computations with the bound  $m_1(s) < A_k$  replaced by  $m_1(s) \leq d$ , then we see that also

$$\mathbb{P}(\exists B \in \Omega_k : X \cap B = \emptyset) < 2^{2kd} \exp(-n 2^{-k-4}).$$

As above, this shows that

$$N(\varepsilon, d) \leq 2^6 d \frac{1 + \log_2(\varepsilon^{-1})}{\varepsilon}$$

for  $\varepsilon \in (0, 1/2)$ . This is, up to constants, the result that was proved recently by Rudolf [15].

*Proof of Lemma 3.* Let  $x$  be uniformly distributed in  $M_k^d$  and let  $B = I_1 \times \dots \times I_d \in \Omega_k$ . Then we know that  $B \in \Omega_k(p, s)$  for some  $p \in M_k^d$  and  $s \in \{1, \dots, 2^k - 1\}^d$ . By the definition of  $\Omega_k(p, s)$ , cf. (1), we know that  $\inf I_\ell < p_\ell$  for all  $\ell = 1, \dots, d$ . On the other hand,

$$\sup I_\ell = \inf I_\ell + |I_\ell| > p_\ell - \frac{1}{2^k} + \frac{s_\ell}{2^k}.$$

We conclude, that  $I_\ell \cap M_k \supset \left\{ p_\ell, \dots, p_\ell + \frac{s_\ell - 1}{2^k} \right\}$  for every  $\ell = 1, \dots, d$ . Therefore, if we set  $B(p, s) := \prod_{\ell=1}^d [p_\ell, p_\ell + \frac{s_\ell - 1}{2^k}]$ , we obtain  $B \cap M_k^d \supset$

$B(p, s) \cap M_k^d$ . Let us also observe, that the probability that a randomly selected element of  $M_k$  falls into  $\{p_\ell, \dots, p_\ell + \frac{s_\ell - 1}{2^k}\}$  is equal to  $\frac{s_\ell}{2^k - 1}$ . Hence,

$$\mathbb{P}(x \in B) \geq \mathbb{P}(x \in B(p, s)) = \prod_{\ell=1}^d \left( \frac{s_\ell}{2^k - 1} \right) = \prod_{\ell \in D_s} \left( \frac{s_\ell}{2^k - 1} \right)$$

with  $D_s := \{\ell \in \{1, \dots, d\} : s_\ell < 2^k - 1\}$ . Next, we use the inequality

$$(3) \quad \left( \frac{j}{2^k - 1} \right) \geq \left( 1 - \frac{1}{k2^k} \right) \left( \frac{j+1}{2^k} \right)^{\frac{k}{k-1}} \quad \text{for all } j = 1, \dots, 2^k - 2,$$

which will be proved later. We obtain

$$\begin{aligned} \mathbb{P}(x \in B) &\geq \left( 1 - \frac{1}{k2^k} \right)^{|D_s|} \prod_{\ell \in D_s} \left( \frac{s_\ell + 1}{2^k} \right)^{\frac{k}{k-1}} \\ &\geq \left( 1 - \frac{1}{k2^k} \right)^{|D_s|} |B|^{\frac{k}{k-1}}. \end{aligned}$$

Since  $|D_s| = m_1(s) < \ln(2)k2^k$  if  $B \in \Omega_k(p, s) \neq \emptyset$ , see (2), we obtain

$$\begin{aligned} \mathbb{P}(x \in B) &\geq \left( 1 - \frac{1}{k2^k} \right)^{\ln(2)k2^k} 2^{-\frac{k^2}{k-1}} \geq \left( 1 - \frac{1}{8} \right)^{8\ln(2)} 2^{-\frac{k^2}{k-1}} \\ &> \frac{1}{4} \cdot 2^{-\frac{k^2}{k-1}} \geq \frac{1}{16} \cdot 2^{-k}, \end{aligned}$$

where we have again used the monotonicity of the sequence  $(1 - 1/k)^k$  and that  $\frac{k^2}{k-1} \leq k + 2$  for  $k \geq 2$ .

To show (3), we prove that

$$(4) \quad \min_{j=1,2,\dots,2^k-2} \frac{j}{(j+1)^{\frac{k}{k-1}}} \geq (2^k - 1) 2^{-\frac{k^2}{k-1}} \left( 1 - \frac{1}{k2^k} \right).$$

As the function  $t \mapsto \frac{t}{(1+t)^{\frac{k}{k-1}}}$  has only one local extremum on  $(0, \infty)$  and this extremum is a local maximum, it is enough to consider  $j \in \{1, 2^k - 2\}$  and to ensure that

$$\min \left( \frac{1}{2^{\frac{k}{k-1}}}, \frac{2^k - 2}{(2^k - 1)^{\frac{k}{k-1}}} \right) \geq (2^k - 1) 2^{-\frac{k^2}{k-1}} \left( 1 - \frac{1}{k2^k} \right).$$

This splits naturally into two inequalities. The first one (for  $j = 1$ ) follows from

$$\frac{1}{2^{\frac{k}{k-1}}} = 2^k \cdot 2^{-\frac{k^2}{k-1}}.$$

The second one (for  $j = 2^k - 2$ ) is equivalent to

$$\frac{2^k - 2}{2^k - 1} \geq \left( \frac{2^k - 1}{2^k} \right)^{\frac{k}{k-1}} \left( 1 - \frac{1}{k2^k} \right),$$

which, by monotonicity, will be established if we prove it with the exponent  $\frac{k}{k-1}$  replaced by 1, i.e.,

$$\frac{2^k - 2}{2^k - 1} \geq \left( \frac{2^k - 1}{2^k} \right) \left( 1 - \frac{1}{k2^k} \right).$$

By simple algebraic manipulations, this is equivalent to

$$2^{2k} - 2 \cdot 2^k \geq (2^k - 1)^2 \left( 1 - \frac{1}{k2^k} \right) = 2^{2k} - 2 \cdot 2^k + 1 - \frac{2^{2k} - 2 \cdot 2^k + 1}{k2^k}$$

and

$$2^{2k} - 2 \cdot 2^k + 1 \geq k \cdot 2^k,$$

which holds for  $k \geq 2$ . This finishes the proof of  $\mathbb{P}(x \in B) > 2^{-k-4}$  for all  $B \in \Omega_k$ .

For the second statement of the lemma note that  $B \cap M_k^d \supset B(p, s) \cap M_k^d$  for all  $B \in \Omega_k(p, s)$ . This shows

$$\mathbb{P}(\forall B \in \Omega_k(p, s): x \in B) \geq \mathbb{P}(x \in B(p, s)) > 2^{-k-4}$$

and therefore

$$\mathbb{P}(\exists B \in \Omega_k(p, s): x \notin B) < 1 - 2^{-k-4} \leq \exp(-2^{-k-4}).$$

□

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INSTITUT FÜR ANALYSIS, JOHANNES KEPLER UNIVERSITÄT, ALTENBERGER STR. 69,  
4040 LINZ, AUSTRIA

*E-mail address:* mario.ullrich@jku.at

DEPT. OF MATHEMATICS FNSPE, CZECH TECHNICAL UNIVERSITY IN PRAGUE, TROJANOVA 13, 12000 PRAGUE, CZECH REPUBLIC

*E-mail address:* jan.vybiral@fjfi.cvut.cz