OPTIMAL SOBOLEV EMBEDDINGS ON $\mathbb{R}^n$

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ABSTRACT. The aim of this paper is to study Sobolev-type embeddings and their optimality. We work in the frame of rearrangement-invariant norms and unbounded domains. We establish the equivalence of a Sobolev embedding to the boundedness of a certain Hardy operator on some cone of positive functions. This Hardy operator is then used to provide optimal domain and range rearrangement-invariant norm in the embedding inequality.

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1. INTRODUCTION

The embeddings of spaces of smooth functions into spaces of integrable functions form a classical part of theory of function spaces. For example the classical Sobolev inequality deals with differentiable functions on bounded domains Ω in $\mathbb{R}^n$, $n \geq 2$, and asserts that, given $1 < p < n$ and setting $q = np/(n-p)$, there exists $C > 0$ such that

$$\Big(\int_\Omega |u(x)|^q \mathrm{d}x\Big)^{1/q} \leq C\Big(\int_\Omega |\nabla u(x)|^p \mathrm{d}x\Big)^{1/p}, \quad u \in C_0^1(\Omega). \tag{1.1}$$

We may combine (1.1) with classical Poincaré inequality for $1 \leq p \leq \infty$,

$$\Big(\int_\Omega |u(x)|^p \mathrm{d}x\Big)^{1/p} \leq C\Big(\int_\Omega |\nabla u(x)|^p \mathrm{d}x\Big)^{1/p}, \quad u \in C_0^1(\Omega). \tag{1.2}$$

Using the standard notation from the theory of function spaces (see Section 2), we may conclude from (1.1) and (1.2)

$$W_p^1(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for} \quad 1 < p < n, \quad q = \frac{np}{n-p}. \tag{1.3}$$

The case $p = n$, called also the limiting case, is of crucial importance. Standard examples show that although $np/(n-p)$ tends to infinity as p tends to n, one may not replace the L^q-norm on the left–hand side of (1.1) by the L^∞-norm.

A comprehensive study of inequalities of Sobolev-type in the frame of the so-called rearrangement-invariant function spaces on bounded domains was carried out in [4]. In this paper we would like to study (1.3) when the bounded domain Ω is replaced by the entire $\mathbb{R}^n$.

Let us briefly outline our approach. Let ϱ_R and ϱ_D be rearrangement-invariant Banach function norms on $(0,\infty)$ (see again Section 2 for a precise definition). Then one may define the following function spaces

$$L^{\varrho_R}(\mathbb{R}^n) = \big\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : ||u|L^{\varrho_R}(\mathbb{R}^n)|| = \varrho_R(u^*(t)) < \infty\big\} \tag{1.4}$$

(1.5)
$$W^1_{\varrho_D}(\mathbb{R}^n) = \big\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : ||u|W^1_{\varrho_D}(\mathbb{R}^n)|| = \varrho_D(u^*(t)) + \varrho_D(|\nabla u|^*(t)) < \infty\big\},$$

where u^* is the non-increasing rearrangement of u.

Then the embedding

$$W^1_{\varrho_D}(\mathbb{R}^n) \hookrightarrow L^{\varrho_R}(\mathbb{R}^n) \tag{1.6}$$

is equivalent to

$$\varrho_R(u^*) \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in W^1_{\varrho_D}(\mathbb{R}^n). \tag{1.7}$$

Using the density arguments we may restrict to $u \in C^1_0(\mathbb{R}^n)$ in (1.7). The inequality (1.7) is the main subject of our study. In Section 3 we reduce this inequality to the boundedness of certain Hardy operator. We are interested in two main questions:

1. Suppose, that the 'range' norm ϱ_R is given. Then we would like to find the norm ϱ_D such that (1.7) holds and, at the same time, it cannot be essentially decreased. Namely, if (1.7) holds with ϱ_D replaced by some norm σ then there exists a constant $C > 0$ such that $\varrho_D(u^*) \le C\sigma(u^*)$ for all functions $u \in C^1_0(\mathbb{R}^n)$.
2. When the 'domain' norm ϱ_D is given, we would like to construct the corresponding optimal 'range' norm ϱ_R. This means that the ϱ_R will have to satisfy two conditions: first, (1.7), and second, that ϱ_R cannot be essentially increased.

In [9] we studied the inequality

$$\varrho_R(u^*) \le c\varrho_D(|\nabla u|^*), \quad u \in W^1_{\varrho_D}(\mathbb{R}^n),$$

which corresponds to one part of (1.6). It turns out that the study of (1.7) requires several new techniques to be developed.

2. Rearrangement-invariant norms

We denote by $\mathfrak{M}(\mathbb{R}^n)$ the set of real-valued Lebesgue-measurable functions on $\mathbb{R}^n$ finite almost everywhere and by $\mathfrak{M}_+(\mathbb{R}^n)$ the class of non-negative functions in $\mathfrak{M}(\mathbb{R}^n)$. Given $f \in \mathfrak{M}(\mathbb{R}^n)$ we define its non-increasing rearrangement by

$$f^*(t) = \inf\{\lambda > 0 : |\{|f(x)| > \lambda\}| \le t\}, \quad 0 < t < \infty. \tag{2.1}$$

For a set $A \subset \mathbb{R}^n$ we denote by $|A|$ its Lebesgue measure. A detailed treatment of rearrangements may be found in [1]. Furthermore we set

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\mathrm{d}s, \quad 0 < t < \infty. \tag{2.2}$$

We point out two subadditivity properties,

$$(f+g)^*(t) \le f^*\left(\frac{t}{2}\right) + g^*\left(\frac{t}{2}\right), \quad 0 < t < \infty, \quad f, g \in \mathfrak{M}(\mathbb{R}^n), \tag{2.3}$$

and

$$(f+g)^{**}(t) \le f^{**}(t) + g^{**}(t), \quad 0 < t < \infty, \quad f, g \in \mathfrak{M}(\mathbb{R}^n). \tag{2.4}$$

We briefly recall some basic aspects of the theory of Banach function norms. For details, see [1].

Definition 2.1. A functional $\varrho : \mathfrak{M}_+(0,\infty) \to [0,\infty]$ is called a *Banach function norm* on $(0,\infty)$ if, for all $f, g, f_n, (n = 1, 2, \dots)$, in $\mathfrak{M}_+(0,\infty)$, for all constants

$a \geq 0$ and for all measurable subsets E of $(0,\infty)$, it satisfies the following axioms

$$
\begin{aligned}
&(A_1) \quad \varrho(f) = 0 \quad \text{if and only if} \quad f = 0 \text{ a.e.};\\
&\qquad\quad \varrho(af) = a\varrho(f);\\
&\qquad\quad \varrho(f+g) \leq \varrho(f) + \varrho(g);\\
&(A_2) \quad \text{if} \quad 0 \leq g \leq f \text{ a.e. then} \quad \varrho(g) \leq \varrho(f);\\
&(A_3) \quad \text{if} \quad 0 \leq f_n \uparrow f \text{ a.e. then} \quad \varrho(f_n) \uparrow \varrho(f);\\
&(A_4) \quad \text{if} \quad |E| < \infty \quad \text{then} \quad \varrho(\chi_E) < \infty;\\
&(A_5) \quad \text{if} \quad |E| < \infty \quad \text{then} \quad \int_E f \leq C_E\,\varrho(f)
\end{aligned}
$$

for some constant $0 < C_E < \infty$, depending on ϱ and E but independent of f.

If, in addition, $\varrho(f) = \varrho(f^*)$, we say ϱ is *rearrangement-invariant* (*r.i.*) *Banach function norm*. We often use the notions *norm* and *r.i. norm* to shorten the notation.

Definition 2.2. The *dilation operator* E_s, $0 < s < \infty$, is defined by

$$(E_s f)(t) = f(st), \quad 0 < t < \infty, \quad f \in \mathfrak{M}(0,\infty). \tag{2.5}$$

The *dual* of a norm ϱ is the functional

$$\varrho'(g) = \sup_{h:\varrho(h)=1} \int_0^1 g(t)h(t)\mathrm{d}t, \quad g,h \in \mathfrak{M}_+(0,\infty). \tag{2.6}$$

Theorem 2.3. (G. H. Hardy, J. E. Littlewood). *If $f,g \in \mathfrak{M}(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq \int_0^\infty f^*(s)g^*(s)\mathrm{d}s. \tag{2.7}$$

Theorem 2.4. (G. G. Lorentz, W. A. J. Luxemburg). *Let ϱ be a Banach function norm. Then*

$$\varrho'' = \varrho. \tag{2.8}$$

Theorem 2.5. (Hardy-Littlewood-Pólya). *Let ϱ be an r.i. norm on $(0,\infty)$ and let $f_1, f_2 \in \mathfrak{M}_0(\mathbb{R}^n)$ with*

$$\int_0^t f_1^*(s)\mathrm{d}s \leq \int_0^t f_2^*(s)\mathrm{d}s, \quad s > 0.$$

Then

$$\varrho(f_1^*) \leq \varrho(f_2^*).$$

Lemma 2.6. (Hardy's Lemma). *Let f_1 and f_2 be non-negative measurable functions on $(0,\infty)$ and suppose*

$$\int_0^t f_1(s)\mathrm{d}s \leq \int_0^t f_2(s)\mathrm{d}s$$

for all $t > 0$. Let h be any non-negative non-increasing function on $(0,\infty)$. Then

$$\int_0^\infty f_1(s)h(s)\mathrm{d}s \leq \int_0^\infty f_2(s)h(s)\mathrm{d}s.$$

3. Reduction to Hardy Operators

In this section we present the main step in the study of (1.7), namely a reduction of (1.7) to the boundedness of certain Hardy operators.

Theorem 3.1. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then the inequality*

$$\varrho_R(u^*) \le c[\varrho_D(u^*) + \varrho_D(|\nabla u|^*)], \quad u \in C_0^1(\mathbb{R}^n), \tag{3.1}$$

holds if and only if there is a constant $K > 0$ such that

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) \le K\varrho_D\left(f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right), \quad f \in \mathfrak{M}_+(0,\infty). \tag{3.2}$$

Proof. Step 1.

Let us suppose that (3.1) holds and that a function $f \in \mathfrak{M}_+(0,\infty)$ is given. We define a new function u in the following way

$$u(x) = \int_{\omega_n|x|^n}^\infty f(t)t^{1/n-1}\mathrm{d}t, \quad x \in \mathbb{R}^n,$$

where ω_n is the volume of unit ball in $\mathbb{R}^n$. Considering level sets of u we obtain

$$u^*(t) = \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s, \;\; |(\nabla u)(x)| = n\omega_n^{1/n}f(\omega_n|x|^n), \;\; |(\nabla u)|^*(t) = n\omega_n^{1/n}f^*(t).$$

Using (3.1) we get finally

$$\varrho_R\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right) = \varrho_R(u^*(t)) \le c\left[\varrho_D(f) + \varrho_D\left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right)\right],$$

which is equivalent to (3.2).

Step 2.

Let us now assume that (3.2) is true and $u \in C_0^1(\mathbb{R}^n)$ is given. Then we use two following observations. The first one is the trivial identity

$$u^*(t) = -\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\mathrm{d}s. \tag{3.3}$$

The second one is the following generalisation of the Pólya—Szegö principle, proved in [3]:

$$\int_0^t \left[-s^{1-1/n}\frac{\mathrm{d}u^*}{\mathrm{d}s}\right]^*(s)\mathrm{d}s \le c\int_0^t |\nabla u|^*(s)\mathrm{d}s, \quad u \in C_0^1(\mathbb{R}^n), \quad t \ge 0. \tag{3.4}$$

Using Theorem of Hardy, Littlewood and Pólya (Theorem 2.5) on (3.4) we obtain

$$\varrho_D\left(-s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\right) \le \varrho_D(|(\nabla u)|^*(t)), \quad u \in C_0^1(\mathbb{R}^n). \tag{3.5}$$

We combine our assumption with these observations.

$$\begin{aligned}
\varrho_R(u^*(t)) &= \varrho_R\left(-\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\mathrm{d}s\right) \\
&\le c\left[\varrho_D\left(-\int_t^\infty \frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\mathrm{d}s\right) + \varrho_D\left(-s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}\right)\right] \\
&\le c\left[\varrho_D(u^*(t)) + \varrho_D(|\nabla u|^*(t))\right].
\end{aligned}$$

We used (3.3), (3.2) with $f = s^{1-1/n}\frac{\mathrm{d}u^*(s)}{\mathrm{d}s}$ and (3.5). ∎

4. Another equivalent version of (1.7)

The inequality (3.2) obtained in Theorem 3.1 is still not suitable for further investigation.

Therefore we derive another equivalent version of (3.1). Namely, we substitute in (3.2)

$$g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s, \quad f \in \mathfrak{M}_+(0,\infty), \quad t > 0. \tag{4.1}$$

We shall need also an inverse substitution. Namely, if g is defined by (4.1), then

$$f(t) = g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \text{ for a.e. } t > 0, \tag{4.2}$$

and

$$\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s = e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u, \text{ for a.e. } t > 0. \tag{4.3}$$

The proof of the first equality is an ordinary differential equation, to prove (4.3) just sum up (4.1) and (4.2). This substitution can now be used to reformulate (3.1). We obtain the following

Theorem 4.1. *Let ϱ_D, ϱ_R be two r.i. Banach function norms on $(0,\infty)$. Then* (3.1) *is equivalent to*

$$\varrho_R\left(e^{nt^{1/n}} \int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \le c\varrho_D(g), \quad g \in \mathbf{G}, \tag{4.4}$$

where $\mathbf{G}$ *is the new class of functions, defined by*

(4.5)

$$\mathbf{G} = \Big\{g \in \mathfrak{M}_+(0,\infty) : \quad \textit{there is a function } f \in \mathfrak{M}_+(0,\infty) \textit{ such that}$$
$$g(t) = f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s \textit{ for all } t > 0\Big\}$$
$$= \left\{g \in \mathfrak{M}_+(0,\infty) : g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \ge 0 \textit{ for all } t > 0\right\}.$$

The proof of this Theorem follows immediately from Theorem 3.1, (4.1), (4.2) and (4.3).

Hence the inequality (3.1) is equivalent to the boundedness of the Hardy-type operator

$$(Gg)(u) = e^{nu^{1/n}} \int_u^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s, \quad u > 0 \tag{4.6}$$

on the set $\mathbf{G}$. Using this notation, we may rewrite (4.3). If g is defined by (4.1), we have $Gg(t) = \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s$. Furthermore, the set $\mathbf{G}$ is the image of the positive cone $\mathfrak{M}_+(0,\infty)$ under the operator

$$f \to f(t) + \int_t^\infty f(s)s^{1/n-1}\mathrm{d}s. \tag{4.7}$$

Before we proceed any further we shall derive some basic properties of the class $\mathbf{G}$.

Remark 4.2. 1. $\mathbf{G}$ contains all non-negative non-increasing functions. To prove this write

(4.8)

$$g(t) - e^{nt^{1/n}} \int_t^\infty g(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \ge g(t)\left\{1 - e^{nt^{1/n}} \int_t^\infty s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\right\} = 0.$$

2. Let g be a function from $\mathbf{G}$ and f be defined by (4.2). Then

$$(Gg)'(t) = \left[e^{nt^{1/n}} \int_t^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u \right]' = -t^{1/n-1} f(t) \le 0. \tag{4.9}$$

To see this, just differentiate (4.3).

Hence the expression on the left-hand side of (4.9) is non-increasing for every function $g \in \mathbf{G}$.

3. The set $\mathbf{G}$ is a *convex cone.* It means that for every two real numbers $\alpha, \beta > 0$ and every two functions $g_1, g_2 \in \mathbf{G}$ we have $\alpha g_1 + \beta \alpha_2 \in \mathbf{G}$. The proof of this statement is trivial.

Remark 4.3. 1. To show some applications we prove that $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-p},p}(\mathbb{R}^n)$ for $1 \le p < n$. In this case, we have $\varrho_R(f) = ||f^*(t)t^{-1/n}||_p$ and $\varrho_D(f) = ||f||_p$. Using (4.9) and the boundedness of classical Hardy operators on L^p we get for every function $g \in \mathbf{G}$

$$\begin{aligned} \varrho_R(Gg) &= ||t^{-1/n}(Gg)^*(t)||_p \\ &= \left|\left| t^{-1/n} e^{nt^{1/n}} \int_t^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u \right|\right|_p \\ &\le \left|\left| t^{-1/n} \int_t^\infty g(u) u^{1/n-1} \mathrm{d}u \right|\right|_p \\ &\le c||t^{-1/n} g(t) t^{1/n}||_p = c||g||_p = c\varrho_D(g). \end{aligned}$$

2. Another application of the obtained results is the embedding $W^1(L^{n,1})(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$. In this case

$$\begin{aligned} \varrho_R(Gg) = \sup_{t>0} (Gg)(t) = (Gg)(0) &= \int_0^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u \\ \le \int_0^\infty g(u) u^{1/n-1} \mathrm{d}u &\le \int_0^\infty g^*(u) u^{1/n-1} \mathrm{d}u = \varrho_D(g) \end{aligned}$$

for every function $g \in \mathbf{G}$. Now we used the property (4.9) and Theorem 2.3.

3. Both these applications recall well-known results. They demonstrate some important aspects of this method. First, the second basic property of the class $\mathbf{G}$, (4.9), lies in the roots of every Sobolev embedding. Second, the boundedness of Hardy operators plays a crucial role in this theory.

4. We haven't used the property (4.8) yet. It will play a crucial role in the study of optimality of obtained results.

5. Optimal domain space

In this section we are going to solve one of our main problems stated in Introduction. We shall construct the optimal domain norm ϱ_D to a given range norm ϱ_R.

We start with a crucial lemma describing one important property of the class $\mathbf{G}$ which shall be useful later on.

Lemma 5.1. *The inequality*

$$\int_t^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u \le c \int_t^\infty g^{**}(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u, \quad t \ge 0. \tag{5.1}$$

holds for every $g \in \mathbf{G}$ with c independent of g.

Proof. We fix $g \in \mathbf{G}$ and $t \ge 0$. Then, according to (4.5), there is a function $f \ge 0$ such that (4.1) holds.

Then the left-hand side of (5.1) can be rewritten as

$$\begin{aligned}
&\int_t^\infty \left(f(u) + \int_u^\infty f(s)s^{1/n-1}\mathrm{d}s \right) u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \\
&= \int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u + \int_t^\infty f(s)s^{1/n-1}\int_t^s u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\mathrm{d}s \qquad (5.2)\\
&= e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s.
\end{aligned}$$

Using (2.4) and Fubini's theorem we get for $g^{**}(u)$

$$\begin{aligned}
g^{**}(u) &\approx f^{**}(u) + \left(\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s\right)^{**}(u) \qquad (5.3)\\
&= f^{**}(u) + \int_u^\infty f(s)s^{1/n-1}\mathrm{d}s + \frac{1}{u}\int_0^u f(s)s^{1/n}\mathrm{d}s.
\end{aligned}$$

The right-hand side of (5.1) is more complicated. We insert the formula (5.3) in (5.1) and use Fubini's theorem. The result looks as follows.

$$\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u \approx \underbrace{\int_t^\infty f^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u}_{\mathrm{I}}$$

$$+\underbrace{\int_t^\infty\int_u^\infty f(s)s^{1/n-1}\mathrm{d}s u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u}_{\mathrm{II}} + \underbrace{\int_t^\infty\int_0^u f(s)s^{1/n-1}\mathrm{d}s u^{1/n-2}e^{-nu^{1/n}}\mathrm{d}u}_{\mathrm{III}}.$$

Each of these three integrals can be further estimated. We start with the second one.

$$\mathrm{II} = e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s - \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s.$$

To deal with integrals I and III we use the notation $h(s) = \int_s^\infty u^{1/n-2}e^{-nu^{1/n}}\mathrm{d}u$. Then

$$\mathrm{I} \geq \int_t^\infty \frac{1}{u}\int_t^u f(s)\mathrm{d}s u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u = \int_t^\infty f(s)h(s)\mathrm{d}s$$

and

$$\mathrm{III} \geq \int_t^\infty\int_t^u f(s)s^{1/n}\mathrm{d}s u^{1/n-2}e^{-nu^{1/n}}\mathrm{d}u = \int_t^\infty f(s)s^{1/n}h(s)\mathrm{d}s.$$

The last three estimates give us

$$\mathrm{I}+\mathrm{II}+\mathrm{III} \geq$$
$$\int_t^\infty f(s)h(s)(s^{1/n}+1)\mathrm{d}s + e^{-nt^{1/n}}\int_t^\infty f(s)s^{1/n-1}\mathrm{d}s - \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s$$

and together with (5.2) we see that it is enough to prove that

$$\int_t^\infty f(s)h(s)(s^{1/n}+1)\mathrm{d}s \geq \int_t^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s.$$

But this inequality is a trivial consequence of the pointwise inequality

$$h(s)(s^{1/n}+1) \geq s^{1/n-1}e^{-ns^{1/n}}, \quad s>0,$$

which may be proved by direct calculation. □

Now we may solve the problem of the optimal domain space.

Theorem 5.2. *Let the norm ϱ_R satisfy*

$$\varrho_R\left(G(g^{**})\right) \le c\varrho_R\left(G(g^{*})\right), \quad g \in \mathfrak{M}_+(0,\infty). \tag{5.4}$$

Then the optimal domain norm ϱ_D corresponding to ϱ_R in the sense described in Introduction is defined by

$$\varrho_D(g) := \varrho_R\left(G(g^{**})\right), \quad g \ge 0. \tag{5.5}$$

Proof. First, we point out that the functional ϱ_D defined by (5.5) is a norm. To verify the axioms $A_1 - A_3$ is trivial. To prove A_4 for ϱ_D we fix a set $E \subset (0,\infty)$ with $|E| < \infty$. Then we get $G\chi_E^*(t) \le \chi_{(0,|E|)}(t)$ for every $t > 0$ and using A_4 for ϱ_R, we get

$$\varrho_D(\chi_E) = \varrho_R(G\chi_E^{**}) \le c\varrho_R(G\chi_E^{*}) \le c\varrho_R(\chi_{(0,|E|)}) < \infty.$$

To verify A_5 for ϱ_D we fix also a set $E \subset (0,\infty)$ with $|E| = a < \infty$ and use A_5 for ϱ_R

$$\begin{aligned}
\varrho_D(g) = \varrho_R(Gg^{**}) &\ge c\int_0^{a/2} (Gg^{**})(t)\mathrm{d}t \\
&\ge c\int_0^{a/2} e^{nt^{1/n}} \int_{a/2}^{a} g^{**}(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\mathrm{d}t \\
&\ge c\, g^{**}(a)\int_0^{a/2} e^{nt^{1/n}}\mathrm{d}t \int_{a/2}^{a} s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s \\
&\ge c_E \int_0^a g^*(s)\mathrm{d}s \ge c_E\int_E g.
\end{aligned}$$

Now we have to verify that (4.4) really holds. Let us fix a $g \in \mathbf{G}$. Then

$$\begin{aligned}
\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) &\le c\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \\
&= c\varrho_D(g).
\end{aligned}$$

(We have used (5.1).)

Finally, we have to show that ϱ_D is optimal. Let us suppose that (4.4) holds with the norm σ instead of ϱ_D. We want to show that $\varrho_D(g) \le c\sigma(g)$ for every function $g \in \mathfrak{M}(0,\infty)$. Using (5.4) and the first property of the class $\mathbf{G}$ from Remark 4.2, namely that $g^* \in \mathbf{G}$ for every function $g \ge 0$, we get

$$\begin{aligned}
\varrho_D(g) &= \varrho_R\left(e^{nt^{1/n}}\int_t^\infty g^{**}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \\
&\le c\varrho_R\left(e^{nt^{1/n}}\int_t^\infty g^{*}(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right) \\
&\le c\sigma(g^*) = c\sigma(g).
\end{aligned}$$

□

6. Optimal Range Space

In this section we solve the converse problem. Namely, the norm ϱ_D is now considered to be fixed and we are searching for the optimal ϱ_R. First of all we shall introduce some notation.

We define

$$Gg(t) = e^{nt^{1/n}} \int_t^\infty g(s) s^{1/n-1} e^{-ns^{1/n}} \mathrm{d}s \tag{6.1}$$

$$Hh(t) = t^{1/n-1} e^{-nt^{1/n}} \int_0^t h(s) e^{ns^{1/n}} \mathrm{d}s \tag{6.2}$$

$$E(s) = e^{-ns^{1/n}} \int_0^s e^{nu^{1/n}} \mathrm{d}u. \tag{6.3}$$

The operators G and H are dual in the following sense.

$$\int_0^\infty h(t) Gg(t) \mathrm{d}t = \int_0^\infty g(u) Hh(u) \mathrm{d}u, \quad g, h \in \mathfrak{M}_+(0,\infty). \tag{6.4}$$

Using similar ideas as in [4] we would like to use duality for definition of ϱ_R. Using this notation, we obtain another equivalent form of (4.4)

$$\sup_{g \in \mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} < \infty. \tag{6.5}$$

We may involve the duality in the following way.

$$\sup_{g \in \mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} = \sup_{g \in \mathbf{G}, h \searrow} \frac{\int (Gg) h}{\varrho_D(g) \varrho_R'(h)} = \sup_{g \in \mathbf{G}, h \searrow} \frac{\int (Hh) g}{\varrho_D(g) \varrho_R'(h)}. \tag{6.6}$$

The supremum is taken over all $g \in \mathbf{G}$ and all non-negative, non-increasing functions h. We have used that Gg is a non-increasing function for every $g \in \mathbf{G}$ (see Remark 4.2) and (6.4). Let us now suppose that taking a supremum over all $g \in \mathfrak{M}_+(0,\infty)$ will give us an equivalent expression. Then we can continue our calculation

$$\sup_{g \in \mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} \approx \sup_{g \in \mathfrak{M}_+(0,\infty), h \searrow} \frac{\int (Hh) g}{\varrho_D(g) \varrho_R'(h)} = \sup_{h \searrow} \frac{\varrho_D'(Hh)}{\varrho_R'(h)} \tag{6.7}$$

and the inequality (4.4) is equivalent to $\varrho_D'(Hh) \leq c \varrho_R'(h)$ for all non-negative, non-increasing functions h. This inequality is already well suited for our needs.

Before we state the main theorem we need to discuss the only weak point in the preceding calculation, namely that place where we came over from a supremum over $g \in \mathbf{G}$ to the supremum over all functions $g \in \mathfrak{M}_+(0,\infty)$.

The answer is given by the following lemma.

Lemma 6.1. *Let us suppose that the norm ϱ_D satisfies the condition*

$$\varrho_D \left(\int_s^\infty f(u) \frac{E(u)}{u} u^{1/n-1} \mathrm{d}u \right) \leq c \varrho_D(f), \quad f \in \mathfrak{M}_+(0,\infty). \tag{6.8}$$

Then

$$\sup_{g \in \mathbf{G}} \frac{\int (Hh) g}{\varrho_D(g)} \approx \sup_{g \in \mathfrak{M}_+(0,\infty)} \frac{\int (Hh) g}{\varrho_D(g)}, \tag{6.9}$$

where the constants of equivalence do not depend on the choice of non-negative, non-increasing function h.

Proof. As $\mathbf{G} \subset \mathfrak{M}_+(0,\infty)$, one inequality in (6.9) follows immediately. To prove the second one, fix a non-negative, non-increasing function h. For every function $f \in \mathfrak{M}_+(0,\infty)$ we define a new function $\tilde{f} \in \mathfrak{M}_+(0,\infty)$ and a function $g(t) = \tilde{f}(t) + \int_t^\infty \tilde{f}(s) s^{1/n-1} \mathrm{d}s$ such that these two conditions are satisfied:

I. $\varrho_D(g) \leq c \varrho_D(f)$,

II. $\int (Hh) g \geq c \int (Hh) f$.

The first condition tells us that the new function g is not too large, the second condition shows that it is not too small either. The way of construction of g ensures that $g \in \mathbf{G}$.

We choose $\tilde{f}(s) = f(s)\frac{E(s)}{s}$, where E is defined by (6.3). Now we prove that by this choice we satisfy both conditions I and II.

In the proof of the first one we use the fact that $s^{-1}E(s) \leq 1$ for all $s \geq 0$. We get

$$\begin{aligned}\varrho_D(g) &= \varrho_D\left(f(s)\frac{E(s)}{s} + \int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right)\\ &\leq \varrho_D\left(f(s)\frac{E(s)}{s}\right) + \varrho_D\left(\int_s^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right)\\ &\leq \varrho_D(f) + c\varrho_D(f) = c\varrho_D(f),\end{aligned}$$

where we used (6.8).

The second condition is more complicated. The left-hand side of the condition II can be simplified by

$$\int (Hh)g = \int (Gg)h = \int_0^\infty h(t)\left(\int_t^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s\right)\mathrm{d}t$$

and the right-hand side by

$$\begin{aligned}\int (Hh)f &= \int_0^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\int_0^u h(t)e^{nt^{1/n}}\mathrm{d}t\mathrm{d}u\\ &= \int_0^\infty h(t)\left(e^{nt^{1/n}}\int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\right)\mathrm{d}t.\end{aligned}$$

We finish the proof by showing that the functions

$$\int_t^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s, \quad e^{nt^{1/n}}\int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u$$

satisfy the assumptions of Hardy's Lemma 2.6 (h is non-increasing). Namely, we want to show that, for every $\xi \geq 0$ and every $f \in \mathfrak{M}_+(0,\infty)$,

$$\int_0^\xi\int_t^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s\mathrm{d}t \geq \int_0^\xi e^{nt^{1/n}}\int_t^\infty f(u)u^{1/n-1}e^{-nu^{1/n}}\mathrm{d}u\mathrm{d}t. \tag{6.10}$$

Using Fubini's Theorem on the right-hand side of (6.10) we get

(6.11)

$$RHS = \int_0^\xi f(s)s^{1/n-1}e^{-ns^{1/n}}\int_0^s e^{nt^{1/n}}\mathrm{d}t\mathrm{d}s + \int_\xi^\infty f(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\int_0^\xi e^{nt^{1/n}}\mathrm{d}t,$$

and using the same Theorem on the left-hand side we obtain

$$\begin{aligned}LHS &= \int_0^\xi \tilde{f}(s)s^{1/n}\mathrm{d}s + \xi\int_\xi^\infty \tilde{f}(s)s^{1/n-1}\mathrm{d}s\\ &= \int_0^\xi f(s)s^{1/n-1}E(s)\mathrm{d}s + \xi\int_\xi^\infty f(s)s^{1/n-2}E(s)\mathrm{d}s.\end{aligned} \tag{6.12}$$

The first integral in the last sum in (6.12) is equal to the first integral in (6.11). So, we shall deal with the second integrals. We shall use the following observation

$$\frac{1}{s}\int_0^s e^{nu^{1/n}}\mathrm{d}u \geq \frac{1}{\xi}\int_0^\xi e^{nu^{1/n}}\mathrm{d}u, \quad s > \xi,$$

and finish the proof by

$$\begin{aligned}
\xi \int_\xi^\infty f(s) s^{1/n-2} E(s) \mathrm{d}s &= \xi \int_\xi^\infty f(s) s^{1/n-2} e^{-ns^{1/n}} \int_0^s e^{nu^{1/n}} \mathrm{d}u \mathrm{d}s \\
&\geq \xi \int_\xi^\infty f(s) s^{1/n-1} e^{-ns^{1/n}} \frac{1}{\xi} \int_0^\xi e^{nu^{1/n}} \mathrm{d}u \mathrm{d}s \\
&= \int_\xi^\infty f(s) s^{1/n-1} e^{-ns^{1/n}} \mathrm{d}s \int_0^\xi e^{nu^{1/n}} \mathrm{d}u.
\end{aligned}$$

□

Equipped with this tool we can now easily solve our problem.

Theorem 6.2. *Let us suppose that the norm ϱ_D satisfies the condition* (6.8) *and that its dual norm satisfies the condition*

$$\varrho_D'(H(h^{**})) \leq c\varrho_D'(H(h^*)), \quad h \in \mathfrak{M}_+(0,\infty). \tag{6.13}$$

Then the optimal range norm in (4.4) *for this ϱ_D is given as a dual norm to $\varrho_D'(H(f^{**}))$. Or, equivalently, the dual of the optimal range norm can be described by $\varrho_R'(f) := \varrho_D'(H(f^{**}))$.*

Proof. According to Lemma 6.1 and the calculation above, the inequality (4.4) is equivalent to

$$\varrho_D'(Hh) \leq c\varrho_R'(h), \quad h \searrow 0. \tag{6.14}$$

But for our choice of ϱ_R' this inequality is trivially true.

To prove the optimality, suppose again, that we have another norm σ, such that (6.14) is true when we substitute its dual norm σ' for the norm ϱ_R'. Then we can estimate

$$\sigma'(f) = \sigma'(f^*) \geq c\varrho_D'(H(f^*)) \geq c\varrho_D'(H(f^{**})) = c\varrho_R'(f), \quad f \in \mathfrak{M}_+(0,\infty).$$

Hence $\sigma(f) \leq c\varrho_R(f)$ and ϱ_R is really optimal.

Finally, we have to prove that the functional $\varrho(f) = \varrho_D'(H(f^{**}))$ is a norm. Again, the axioms $A_1 - A_3$ are trivially satisfied. Using (6.13), Hardy's Lemma 2.6 and axiom A_4 for ϱ_D' we get also A_4 for ϱ. A_5 follows from the same axiom for ϱ_D'. □

7. The study of (5.4) and (6.13)

In this section we derive conditions sufficient for (6.8) and (6.13) to hold. In general we follow the idea of [4], Theorem 4.4. First of all, we define the dilation operator E. For every function $f \in \mathfrak{M}_+(0,\infty)$, we define

$$(E_s f)(t) = f(st), \quad t > 0, \quad s > 0.$$

It is very well known, that for every r.i. norm ϱ on $\mathfrak{M}_+(0,\infty)$ and every $s > 0$ the operator E_s satisfies the inequality

$$\varrho(E_s f) \leq c\varrho(f), \quad f \in \mathfrak{M}_+(0,\infty).$$

The smallest possible constant c in this inequality (which depends of course on s) is denoted by $h_\varrho(s)$. Hence

$$h_\varrho(s) = \sup_{f \not\equiv 0} \frac{\varrho(E_s f)}{\varrho(f)}.$$

Now we are ready to prove our first result.

Theorem 7.1. *If a rearrangement–invariant norm ϱ_R satisfies $\int_0^1 s^{-1/n} h_{\varrho_R}(s) \mathrm{d}s < \infty$, then it also satisfies the condition* (5.4).

Proof. Step 1.
Let us suppose that the positive real numbers s,t,y satisfy $st<y$ and $0<s<1$. Then $t^{1/n}<(y/s)^{1/n}$ and, consequently,

$$e^{nt^{1/n}-n(y/s)^{1/n}} \le \left[e^{nt^{1/n}-n(y/s)^{1/n}}\right]^{s^{1/n}} = e^{n(st)^{1/n}-ny^{1/n}}.$$

So, for every function $f\in\mathfrak{M}_+(0,\infty)$, we obtain

$$e^{nt^{1/n}}\int_{st}^{\infty} f^*(y)y^{1/n-1}e^{-n(y/s)^{1/n}}\mathrm{d}y \le e^{n(st)^{1/n}}\int_{st}^{\infty} f^*(y)y^{1/n-1}e^{-ny^{1/n}}\mathrm{d}y.$$

Step 2.
We may now come to the proof of the Theorem. Fix a function $g\in\mathfrak{M}_+(0,\infty)$, with $\varrho_R'(g)=1$. Then we use several times Fubini's Theorem, the change of variables, and inequality from Step 1, and obtain

$$\begin{aligned}\int_0^\infty g^*(t)Gf^{**}(t)\mathrm{d}t &= \int_0^\infty g^*(t)e^{nt^{1/n}}\int_t^\infty f^{**}(s)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\mathrm{d}t\\
&= \int_0^\infty s^{1/n-1}e^{-ns^{1/n}}\int_0^s g^*(u)e^{nu^{1/n}}\mathrm{d}u\int_0^1 f^*(st)\mathrm{d}t\mathrm{d}s\\
&= \int_0^1\int_0^\infty f^*(st)s^{1/n-1}e^{-ns^{1/n}}\int_0^s g^*(u)e^{nu^{1/n}}\mathrm{d}u\mathrm{d}s\mathrm{d}t\\
&= \int_0^1\int_0^\infty g^*(u)e^{nu^{1/n}}\int_u^\infty f^*(st)s^{1/n-1}e^{-ns^{1/n}}\mathrm{d}s\mathrm{d}u\mathrm{d}t\\
&= \int_0^1 t^{-1/n}\int_0^\infty g^*(u)e^{nu^{1/n}}\int_{tu}^\infty f^*(y)y^{1/n-1}e^{-n(y/t)^{1/n}}\mathrm{d}y\mathrm{d}u\mathrm{d}t\\
&= \int_0^1 s^{-1/n}\int_0^\infty g^*(t)e^{nt^{1/n}}\int_{st}^\infty f^*(y)y^{1/n-1}e^{-n(y/s)^{1/n}}\mathrm{d}y\mathrm{d}t\mathrm{d}s\\
&\le \int_0^1 s^{-1/n}\int_0^\infty g^*(t)e^{n(st)^{1/n}}\int_{st} f^*(y)y^{1/n-1}e^{-ny^{1/n}}\mathrm{d}y\mathrm{d}t\mathrm{d}s\\
&= \int_0^1 s^{-1/n}\int_0^\infty g^*(t)(Gf^*)(st)\mathrm{d}t\mathrm{d}s.\end{aligned}$$

Taking a supremum over g, we obtain that the left-hand side of (5.4) can be estimated from above by

$$\begin{aligned}\sup_{g\ge 0:\varrho_R'(g)=1}\int_0^1 s^{-1/n}\int_0^\infty g^*(t)(Gf^*)(st)\mathrm{d}t\mathrm{d}s\\
&\le \int_0^1 s^{-1/n}\varrho_R((Gf^*)(s\cdot))\mathrm{d}s\\
&\le \int_0^1 s^{-1/n}h_{\varrho_R}(s)\varrho_R(Gf^*)\mathrm{d}s\\
&= \left(\int_0^1 s^{-1/n}h_{\varrho_R}(s)\mathrm{d}s\right)\varrho_R(Gf^*).\end{aligned}$$

□

Very similar theorem can be derived also for inequality (6.13).

Theorem 7.2. *If rearrangement–invariant norm σ satisfies $\int_0^1 s^{-1/n}h_\sigma(s)\mathrm{d}s<\infty$ then it satisfies also* (6.13).

Proof. We follow the same idea as in the previous proof.

Step 1. Let us suppose that positive real numbers u, s, t satisfy $0 < s < 1$, $u < st$. Then $(u/s)^{1/n} - t^{1/n} < 0$ and

$$e^{n(u/s)^{1/n} - nt^{1/n}} \le \left[e^{n(u/s)^{1/n} - nt^{1/n}}\right]^{s^{1/n}} = e^{nu^{1/n} - n(st)^{1/n}}.$$

Finally, we obtain for these u, s, t and every function $h \in \mathfrak{M}_+(0,\infty)$

$$e^{-nt^{1/n}} \int_0^{st} h^*(u) e^{n(u/s)^{1/n}} \mathrm{d}u \le e^{-n(st)^{1/n}} \int_0^{st} h^*(u) e^{nu^{1/n}} \mathrm{d}u.$$

Step 2.

Let us again fix $g \in \mathfrak{M}_+(0,\infty)$ with $\sigma'(g) = 1$. We can do similar estimates as above.

$$\begin{aligned}
&\int_0^\infty g(u) H(h^{**})(u) \mathrm{d}u \\
&\quad = \int_0^\infty e^{nt^{1/n}} \int_t^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u \frac{1}{t} \int_0^t h^*(s) \mathrm{d}s \mathrm{d}t \\
&\quad = \int_0^\infty e^{nt^{1/n}} \int_t^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u \int_0^1 h^*(tv) \mathrm{d}v \mathrm{d}t \\
&\quad = \int_0^1 \int_0^\infty h^*(tv) e^{nt^{1/n}} \int_t^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \mathrm{d}u \mathrm{d}t \mathrm{d}v \\
&\quad = \int_0^1 \int_0^\infty g(u) u^{1/n-1} e^{-nu^{1/n}} \int_0^u h^*(tv) e^{nt^{1/n}} \mathrm{d}t \mathrm{d}u \mathrm{d}v \\
&\quad = \int_0^1 s^{-1} \int_0^\infty g(t) t^{1/n-1} e^{-nt^{1/n}} \int_0^{st} h^*(u) e^{n(u/s)^{1/n}} \mathrm{d}u \mathrm{d}t \mathrm{d}s \\
&\quad \le \int_0^1 s^{-1} \int_0^\infty g(t) t^{1/n-1} e^{-n(st)^{1/n}} \int_0^{st} h^*(u) e^{nu^{1/n}} \mathrm{d}u \mathrm{d}t \mathrm{d}s \\
&\quad = \int_0^1 s^{-1/n} \int_0^\infty g(t) H(h^*)(st) \mathrm{d}t \mathrm{d}s.
\end{aligned}$$

Now we take again supremum over g with $\sigma'(g) = 1$ and obtain

$$\sigma(H(h^{**})) \le \int_0^1 s^{-1/n} \sigma(E_s(H(h^*))) \mathrm{d}s \le \left(\int_0^1 s^{-1/n} h_\sigma(s) \mathrm{d}s\right) \sigma(H(h^*)).$$

□

8. The limiting embedding

In this section we consider the case of limiting Sobolev embedding, where ϱ_D is set to be $\varrho_D(f) = \varrho_n(f) = (\int |f|^n)^{1/n}$. In that case, $\varrho'_D(f) = \varrho_{n'}(f)$, where n' is the conjugated exponent to n, namely $\frac{1}{n} + \frac{1}{n'} = 1$. Direct calculation shows that $h_{\varrho'_D}(s) = s^{-1/n'}$ and $\int_0^1 s^{-1/n} h_{\varrho'_D}(s) \mathrm{d}s = \infty$. It means that we may not use Theorem 7.2 to verify the condition (6.13). Standard examples ($h(s) = \frac{1}{s|\log s|^2}\chi_{(0,1/2)}(s)$) show that (6.13) is not satisfied, hence even some improved version of Theorem 7.2 could not help.

To include this important case into the frame of our work, we develop a finer theory of optimal range space. This is described in the following

Theorem 8.1. *Let ϱ_D be a given r.i. norm such that* (6.8) *holds and*

$$\varrho'_D(H\chi_{(0,1)}) < \infty. \tag{8.1}$$

Then we set

$$\sigma(h) = \varrho_D'(Hh^*), \quad h \in \mathfrak{M}(0,\infty),$$

and claim that

$$\varrho_R = \sigma'$$

is a norm, which satisfies (4.4) *and, with a domain norm* ϱ_D *fixed, is optimal for* (4.4) *to hold.*

Proof. Step 1.

We prove that ϱ_R is a norm. The axioms A_2 and A_3 are easy to verify. Let us assume that $\varrho_R(f) = 0$ for some $f \in \mathfrak{M}(0,\infty)$. Then

$$\sup_{\sigma(g)=1} \int fg = 0.$$

But, according to (8.1), $\sigma(\chi_E)$ is finite for every measurable set $E \subset (0,\infty)$ with finite measure. Hence $\int_E f = 0$ for every such set and $f = 0$ almost everywhere. This proves A_1.

To verify A_5 for ϱ_R we fix a set $E \subset (0,\infty)$ with $|E| < \infty$ and write for every function $f \in \mathfrak{M}_+(0,\infty)$

$$\varrho_R(f) = \sup_{\sigma(h)\neq 0} \frac{\int fh}{\sigma(h)} \geq \frac{\int f\chi_E}{\sigma(\chi_E)} = c_E \int_E f.$$

The axiom A_4 is an easy consequence of the estimate

$$\sigma(g) \geq c_E \int_0^{|E|} g^*(u)\mathrm{d}u, \quad g \in \mathfrak{M}_+(0,\infty). \tag{8.2}$$

To prove (8.2), we write

$$\begin{aligned}
\sigma(g) = \varrho_D'(Hg^*) &= \varrho_D'\left(t^{1/n-1}e^{-nt^{1/n}} \int_0^t g^*(u)e^{nu^{1/n}}\mathrm{d}u\right) \\
&\geq \frac{\int_0^{2|E|} t^{1/n-1}e^{-nt^{1/n}} \int_0^t g^*(u)e^{nu^{1/n}}\mathrm{d}u\mathrm{d}t}{\varrho_D(\chi_{(0,2|E|)})} \\
&\geq c\int_0^{2|E|} g^*(u)e^{nu^{1/n}} \int_u^{2|E|} t^{1/n-1}e^{-nt^{1/n}}\mathrm{d}t\mathrm{d}u \\
&\geq c_E \int_0^{|E|} g^*(u)\mathrm{d}u.
\end{aligned}$$

Step 2.

We show that ϱ_R and ϱ_D satisfy (4.4). We proceed similarly as in Section 6 and write

$$\sup_{g\in\mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} = \sup_{g\in\mathbf{G}} \frac{\sigma'(Gg)}{\varrho_D(g)} = \sup_{g\in\mathbf{G},h\searrow} \frac{\int(Gg)h}{\varrho_D(g)\sigma(h)} = \sup_{g\in\mathbf{G},h\searrow} \frac{\int(Hh)g}{\varrho_D(g)\sigma(h)}. \tag{8.3}$$

We have used the definition of ϱ_R in the first equality, (4.9) in the second one, and (6.4) in the last one.

According to Lemma 6.1 we may continue in (8.3) and get

$$\begin{aligned}
\sup_{g\in\mathbf{G}} \frac{\varrho_R(Gg)}{\varrho_D(g)} &= \sup_{h\searrow} \frac{1}{\sigma(h)} \sup_{g\in\mathbf{G}} \frac{\int(Hh)g}{\varrho_D(g)} \\
&\approx \sup_{h\searrow} \frac{1}{\sigma(h)} \sup_{g\in\mathfrak{M}_+(0,\infty)} \frac{\int(Hh)g}{\varrho_D(g)} = \sup_{h\searrow} \frac{\varrho_D'(Hh^*)}{\sigma(h)} = 1.
\end{aligned}$$

Step 3.

Finally, we prove the optimality of ϱ_R. Let the Banach function norms ν and ϱ_D satisfy (4.4) with ν instead of ϱ_R, namely let

$$\sup_{g\in\mathbf{G}} \frac{\nu(Gg)}{\varrho_D(g)} < \infty.$$

Then we get

$$\infty > \sup_{g\in\mathbf{G}} \frac{\nu(Gg)}{\varrho_D(g)} = \sup_{g\in\mathbf{G},h\searrow} \frac{\int (Gg)h}{\varrho_D(g)\nu'(h)} \approx \sup_{h\searrow} \frac{\varrho'_D(Hh^*)}{\nu'(h)}.$$

We used again Lemma 6.1. Hence, for every $h \in \mathfrak{M}_+(0,\infty)$, we get

$$\sigma(h) = \varrho'_D(Hh^*) \le c\,\nu'(h)$$

and, consequently, we get for every $f \in \mathfrak{M}_+(0,\infty)$

$$\nu(f) = \nu''(f) \le c\,\sigma'(f) = c\,\varrho_R(f).$$

So the norm ν is (up to some constant) smaller than the norm ϱ_R. □

Let us apply Theorem 8.1 to the limiting Sobolev embeddings with

$$\varrho_D(f) = \varrho_n(f) = (\int |f|^n)^{1/n}$$

or

$$\varrho_D(f) = \varrho_{n,1}(f) = \int_0^\infty t^{1/n-1} f^*(t)\mathrm{d}t,$$

respectively. Direct calculation shows that (8.1) is satisfied in both these cases.
To verify (6.8), we point out that

$$E(s) \approx \begin{cases} s, & \text{for} \quad s \in (0,1), \\ s^{1-1/n}, & \text{for} \quad s \in (1,\infty). \end{cases} \tag{8.4}$$

Hence, by Fubini's theorem, (8.4) and Lemma 2.6,

$$\begin{aligned} \varrho_{n,1}\left(\int_t^\infty f(u)\frac{E(u)}{u}u^{1/n-1}\mathrm{d}u\right) &= n\int_0^\infty f(u)\frac{E(u)}{u}u^{1/n-1}u^{1/n}\mathrm{d}u \\ &\le c\int_0^\infty t^{1/n-1}f(t)\mathrm{d}t \le \int_0^\infty t^{1/n-1}f^*(t)\mathrm{d}t = c\varrho_{n,1}(f). \end{aligned}$$

In the first case $\varrho_D = \varrho_n$, (6.8) is a consequence of Hardy's inequality. We refer to [7] for details. So, in both the cases, Theorem 8.1 is applicable and gives the optimal range norm.
Let us give now a precise characterisation of this norm.

Theorem 8.2. *Let $\varrho_D = \varrho_n$. Then, the optimal range norm, ϱ_R, satisfies*

$$\varrho_R(f) \approx \varrho_n(f) + \lambda(f^*\chi_{(0,1)}), \tag{8.5}$$

where

$$\lambda(g) := \left(\int_0^1 \left(\frac{g^*(t)}{\log(\frac{e}{t})}\right)^n \frac{\mathrm{d}t}{t}\right)^{\frac{1}{n}}, \qquad g \in \mathfrak{M}(0,1).$$

Proof. We first recall that for $\varrho_D = \varrho_n$, both (6.8) and (8.1) are satisfied. Thus, by Theorem 8.1,

$$\begin{aligned}
\varrho_R'(h) \approx \varrho_{n'}(Hh^*) &= \varrho_{n'}\left(t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)\\
&\approx \varrho_{n'}\left(\chi_{(0,1)}(t)t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)\\
&\quad + \varrho_{n'}\left(\chi_{(1,\infty)}(t)t^{1/n-1}e^{-nt^{1/n}}\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)\\
&=: I + II.
\end{aligned}$$

Since, for $0 \le s \le t \le 1$, we have

$$e^{-n} \le e^{n(s^{1/n}-t^{1/n})} \le 1,$$

we obtain

$$I \approx \varrho_{n'}\left(\chi_{(0,1)}(t)t^{1/n-1}\int_0^t h^*(s)\,\mathrm{d}s\right) = \left(\int_0^1\left(\int_0^t h^*(s)\,\mathrm{d}s\right)^{n'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}.$$

As for II, we have

$$\begin{aligned}
II &= \left(\int_1^\infty\left(\int_0^t h^*(s)e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\ge \left(\int_1^\infty h^*(t)^{n'}\left(e^{-nt^{1/n}}\int_0^t e^{ns^{1/n}}\,\mathrm{d}s\right)^{n'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\approx \left(\int_1^\infty h^*(t)^{n'}\left(t^{1-1/n}\right)^{n'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}} \qquad \text{by (8.4)}\\
&= \left(\int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}.
\end{aligned}$$

Conversely, by the weighted Hardy inequality (cf. [7]),

$$\begin{aligned}
II &\approx \left(\int_1^\infty\left(\int_0^1 h^*(s)e^{ns^{1/n}}\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\quad + \left(\int_1^\infty\left(\int_1^t h^*(s)e^{ns^{1/n}}\mathrm{d}s\right)^{n'} e^{-nn't^{1/n}}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}}\\
&\le c\left[\int_0^1 h^*(s)\mathrm{d}s + \left(\int_1^\infty h^*(t)^{n'}\mathrm{d}t\right)^{\frac{1}{n'}}\right]\\
&\le c\left[\left(\int_0^1\left(\int_0^t h^*(s)\mathrm{d}s\right)^{n'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{n'}} + \left(\int_1^\infty h^*(t)^{n'}\mathrm{d}t\right)^{\frac{1}{n'}}\right]
\end{aligned}$$

Altogether,

$$\varrho_R'(g) \approx \left(\int_0^1\left(\int_0^t h^*(s)\,\mathrm{d}s\right)^{n'}\frac{\mathrm{d}t}{t} + \int_1^\infty h^*(t)^{n'}\,\mathrm{d}t\right)^{\frac{1}{n'}}.$$

Now, set

$$\nu(g) := \left(\int_0^\infty g^*(t)^n v(t)\,\mathrm{d}t\right)^{\frac{1}{n}},$$

where

$$v(t) = \begin{cases} t^{-1}\left(\log \frac{e}{t}\right)^{-n}, & t \in (0,1), \\ 1, & t \in (1,\infty). \end{cases}$$

Then, by [8, Theorem 4], ν is an r.i. norm. More precisely, it is a special case of a classical Lorentz norm whose Köthe dual has been characterised in [8, Theorem 1]. Thus,

$$\nu'(f) \approx \left(\int_0^\infty \left(\int_0^t f^*(s)\,\mathrm{d}s \right)^{n'} \frac{v(t)}{\left(\int_0^t v(s)\,\mathrm{d}s\right)^{n'}}\,\mathrm{d}t \right)^{\frac{1}{n'}} \approx \varrho_R'(f),$$

as an easy calculation shows.

Finally, since both ν and ϱ_R are r.i. norms, it follows from the Principle of Duality (2.8) that

$$\varrho_R \approx \nu,$$

as desired. □

Remark 8.3. We note that λ from Theorem 8.2 is the well-known norm discovered in various contexts independently by Maz'ya [6], Hanson [5] and Brézis–Wainger [2].

Finally, we apply Theorem 5.2 to find the optimal domain norm ϱ_D to a given range norm

$$\varrho_R(f) = \varrho_\infty(f) = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} |f(x)|.$$

According to Theorem 7.1, (5.4) is satisfied and the optimal domain norm is given by

$$\varrho_D(f) \approx \sup_{t>0} (Gf^*)(t) = \int_0^\infty f^*(s) s^{1/n-1} e^{-ns^{1/n}}\mathrm{d}s, \quad f \in \mathfrak{M}(\mathbb{R}^n),$$

and is essentially smaller than the norm $\varrho_{n,1}(f) = \int_0^\infty t^{1/n-1} f^*(t)\mathrm{d}t$. This improves the second example from Remark 4.3. We have used (4.9). Easy calculation shows a direct analogy to (8.5)

$$\varrho_D(f) \approx f^*(1) + \int_0^1 f^*(t)t^{1/n-1}\mathrm{d}t \approx \varrho_\infty(f^*\chi_{(1,\infty)}) + \varrho_{n,1}(f^*\chi_{(0,1)}), \quad f \in \mathfrak{M}(\mathbb{R}^n).$$

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