Electric Fields in Matter

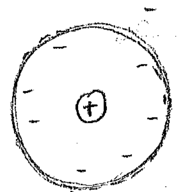
- We will study how electric fields behave in matter.
- Most of the material belong to one of two large classes
  - i) Conductors $\longrightarrow$ we have already studied how E-field behave in conductor.
  - ii) Insulators (Dielectrics) : All charges are attached to specific atoms.

$\longrightarrow$ Electric field still affect Dielectrics regardless all charges are binded to an atom.

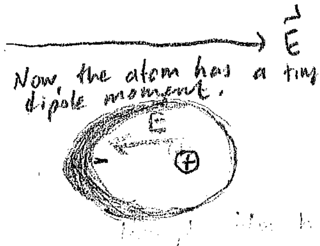
☆ There are 2 mechanisms that E-fields can distort the charge distribution of a dielectric atom or molecule ; stretching and rotating.

Induced dipole

What happens when a neutral atom is placed in an electric field?



Without $\vec{E}$, e^-'s are distributed around nucleus equally



It is quite likely to find the e^-'s at left than at right.

☆ Streching ends when the E created by the atom is equal to the external E-field.

☆ Sometimes, if $\vec{E}$ is very strong, the atom can be broken apart (ionization).

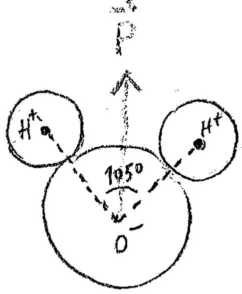
☆ If we place a neutral atom in an electric field it will have a small dipole moment pointing in the same direction as $\vec{E}$.

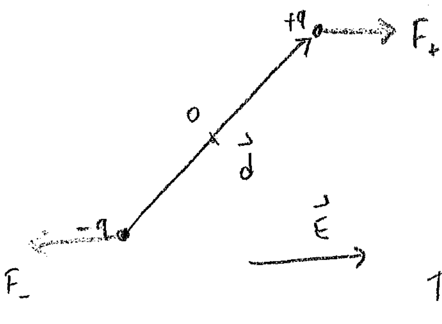
$$\vec{p} = \alpha \vec{E}$$

α $\rightarrow$ atomic polarizability. It depends on the structure of the atom, and it is determined experimentally.

Alignment of Polar Moleccules (Dipol moments)

Some molecules have permanent dipole moments. How do these molecules behave when we put them in an external $\vec{E}$-field?





If $\vec{E}$ is uniform F_+ will cancel F_-. But still there will be a torque

$$\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$

$$= \left[\frac{\vec{d}}{2} \times q\vec{E}\right] + \left[\left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E})\right] = q\,\vec{d} \times \vec{E}$$

Thus a dipole in a uniform E-field experiences a torque

$$\boxed{\vec{N} = \vec{p} \times \vec{E}}$$

• If the field is not uniform, there will be a net force on the dipole; $\vec{F}_+$ wouldn't balance $\vec{F}_-$.

$$\vec{F} = \vec{F}_+ - \vec{F}_- = q(\vec{E}_+ - \vec{E}_-) = q\,\Delta\vec{E}$$

$\Delta\vec{E}$: the difference between the field at the plus end and the field at the minus end.

• By assuming that dipole is very short:

$$\Delta E_x \cong \vec{\nabla} E_x \cdot \vec{d} = \frac{\partial E_x}{\partial x} d_x + \frac{\partial E_x}{\partial y} d_y + \frac{\partial E_x}{\partial z} d_z$$
$$\Delta E_y \cong \vec{\nabla} E_y \cdot \vec{d} = \frac{\partial E_y}{\partial x} d_x + \frac{\partial E_y}{\partial y} d_y + \frac{\partial E_y}{\partial z} d_z$$
$$\Delta E_z \cong \vec{\nabla} E_z \cdot \vec{d} = \frac{\partial E_z}{\partial x} d_x + \frac{\partial E_z}{\partial y} d_y + \frac{\partial E_z}{\partial z} d_z$$

$$\begin{aligned}\Delta\vec{E} &= \Delta E_x \hat{x} + \Delta E_y \hat{y} + \Delta E_z \hat{z} \\ &= (\vec{\nabla}\cdot\vec{d}) E_x \hat{x} + (\vec{\nabla}\cdot\vec{d}) E_y \hat{y} + (\vec{\nabla}\cdot\vec{d}) E_z \hat{z} \\ &= (\vec{\nabla}\cdot\vec{d}) \{E_x\hat{x} + E_y\hat{y} + E_z\hat{z}\} = (\vec{\nabla}\cdot\vec{d})\vec{E}\end{aligned}$$

$$\Rightarrow \vec{F} = q(\vec{\nabla}\cdot\vec{d})\vec{E} = \underbrace{\underbrace{(\vec{\nabla}\cdot\vec{p})}_{\text{scaler}}\vec{E}}_{\text{vectoral}}$$

__Polarization__ Instead of individual atoms and molecules;

• What happens to a piece of dielectric material when it is placed in an E-field?

• If the material consists of neutral atoms (non polar molecules)

→ The field will induce in each a tiny dipole moment

If the material is made up of polar molecules.

→ Each permanent dipole will experience a torque tending to line it up along the field direction.

⇒ These two mechanisms produce the same result:

★ A lot of tiny dipoles pointing along the direction of the field.

⇒ The material becomes "polarized".

⇝ A convenient measure for this effect

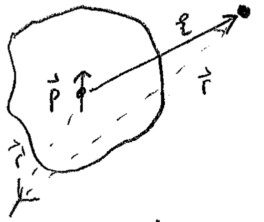
$\vec{P}$ = dipole moment per unit volume. or simply called __POLARIZATION__

★ From now on we will not consider how this polarization happened. We will simply forget about the cause of polarization and study the field that a group of polarized material itself produces.

→ We will put everything together later; so that we will both consider the field causing $\vec{P}$ and the new field due to $\vec{P}$.

The Field of A Polarized Object

We have a polarized material (object with a lot of microscopic dipoles lined up)
Dipole moment per unit of volume $\vec{P}$ is given. $\left\{ \text{units of } \vec{P} \quad \frac{Cm}{m^3} \quad C/m^2 \right\}$
→ what is the field produced by this object?



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{\mathscr{r}} \cdot \vec{p}}{r^2} \quad \text{from a single dipole.}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\hat{\mathscr{r}} \cdot \vec{P}(\vec{r}\,')}{\mathscr{r}^2} d\tau' \qquad (\vec{\mathscr{r}} = \vec{r} - \vec{r}\,')$$

In spherical coordinates:

$$\vec{\nabla} t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi} \Rightarrow \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

→ prime represents it is a differentiation w.r.t. source coordinates.

$$\boxed{\frac{\partial}{\partial r'}\left(\frac{1}{r - r'}\right) = \frac{\partial}{\partial r'}(r-r')^{-1} = (-1)\frac{1}{(r-r')^2}(-1) = \frac{1}{(r-r')^2} \equiv \frac{1}{R^2}}$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \vec{P} \cdot \vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

If we consider: $\vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\vec{\nabla}' \cdot \vec{P}) + \vec{P} \cdot \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right)$

$$\Rightarrow \vec{P} \cdot \vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = \vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}(\vec{\nabla}' \cdot \vec{P})$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \left[\vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}} \vec{\nabla}' \cdot \vec{P}\right] d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\int_V \vec{\nabla}' \cdot \left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau' - \int_V \frac{1}{\mathscr{r}}(\vec{\nabla}' \cdot \vec{P}) d\tau'\right]$$

• Using divergence theorem: $\int (\vec{\nabla} \cdot \vec{A}) d\tau = \oint \vec{A} \cdot d\vec{a}$

$$V(\vec{r}) = \underbrace{\frac{1}{4\pi\epsilon_0} \oint_S \frac{1}{\mathscr{r}} \vec{P} \cdot d\vec{a}'}_{\text{surface charge}} - \underbrace{\frac{1}{4\pi\epsilon_0} \int_V \frac{1}{\mathscr{r}} (\vec{\nabla}' \cdot \vec{P}) d\tau'}_{\text{volume charge}}$$

must be related with a surface charge density $\sigma_b \equiv \vec{P} \cdot \hat{n}$

must be related with a volume charge density $\rho_b \equiv -\vec{\nabla} \cdot \vec{P}$

$\hat{n}$: unit vector normal to surface

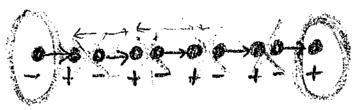
remember $\vec{P}$ is in units of C/m^2 (and field)

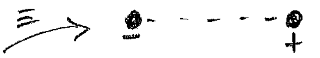
$$\Rightarrow \boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_b}{\mathscr{r}} da' + \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_b}{\mathscr{r}} d\tau}$$

★ Potential of a polarized object is the same as that produced by a volume charge density $\rho_b = -\vec{\nabla} \cdot \vec{P}$ plus surface charge density $\sigma_b = \vec{P} \cdot \hat{n}$. Instead of integrating the contribution of individual dipoles, we simply find the bound charges and then calculate the field they produce, in the same way we calculate the field of other volume or surface charges.

Physical Interpretation of Bound Charges.

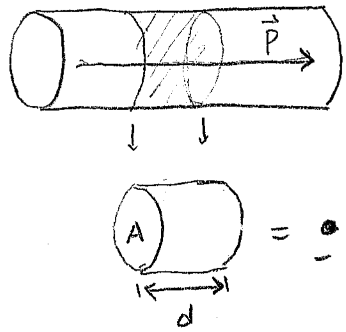
- ρ_b and σ_b are actually physical accumulations of charge.
- Consider a long strip of dipoles.





as if we removed an electron from the end and put it to the beginning

- Actually no e^- is moved all the way down to the other end but small displacement add up and yield one large one in the end.
- To calculate the actual amount of bound charge consider

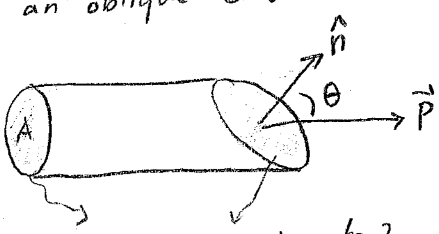


$\rightsquigarrow$ Dipole moment of this segment is $\rightarrow P(Ad)$

and bound charge piles up at the end of the segment $q = PA$

Therefore $\boxed{\sigma_b = \frac{q}{A} = P}$ $\rightarrow$ This holds when we cut the slice perpendicular

For an oblique cut :



The same charge has to be accumulated.

Charge must be the same but

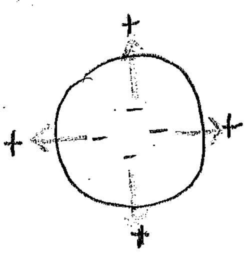
$A = A_{end} \cos\theta$

$$\sigma_b = \frac{q}{A_{end}} = \frac{PA}{\frac{A}{\cos\theta}} = P\cos\theta = \vec{P}\cdot\hat{n}$$

$\rightarrow$ we confirmed our previous result

- If the polarization is nonuniform, we get accumulation of bound charge within the material as well as on the surface.

Think of a small volume where there is a net $\vec{P}$ flux out.



- It suggests that diverging $\vec{P}$ results in a pileup of negative charge.
- Net bound charge $\int \rho_b \, d\tau$ in a given volume is equal and opposite to the amount has been pushed out through the surface.

$$\underset{\text{equal and opposite}}{+\int_{\mathcal{V}} \rho_b \, d\tau = -\oint_S \underbrace{\vec{P}\cdot d\vec{a}}_{\text{unit: C}}} = \underbrace{-\int_{\mathcal{V}} (\vec{\nabla}\cdot\vec{P}) \, d\tau}_{\text{Divergence theorem.}} \Rightarrow \rho_b = -\vec{\nabla}\cdot\vec{P}$$

$\rightarrow$ again we have confirmed our previous result

• Also keep in mind that

$\rho_b = -\vec{\nabla} \cdot \vec{P}$: if divergence is positive then it means that some net $\vec{P}$ must come out from the infinitesimal volumes where ρ_b is located. So as the expression implies ρ_b must be negative as expected. (or vice versa).

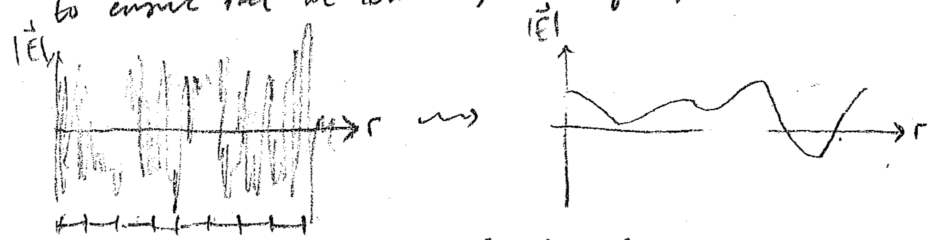
The Field Inside a Dielectric

• A real dielectric consists of physical dipoles.
• But we derived σ_b and ρ_b starting with $V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{\imath} \cdot \vec{p}}{\imath^2}$
• Moreover, we described discrete physical dipoles with a continuous density function $\vec{P}$.
  ★ How are we going to justify this?

i) Outside the dielectric there is no problem. $\imath$ is many times greater than separation distance between + and −. Dipole contribution dominates overwhelmingly.

ii) Inside the dielectric, electric field is truly complicated. and probably it is impossible to completely calculate it.

★ So the strategy to consider the field inside a dielectric is to average the field over regions large enough to contain many thousands of atoms so that the uninteresting microscopic fluctuations are smoothed over, and yet small enough to ensure that we don't ignore any significant large variations in the field



Average over regions much smaller than the dimensions of the object.

★ So, when we talk about "field inside matter" we will mean a macroscopic field.

• So, let's justify this point:

Gauss Law in the Presence of Dielectrics

- we will put together bound charges and everything else (free charge)
- The free charge might consist of e^- on a conductor or ions in the dielectric material or anything else. (everything else that is not a result of polarization)
- Within the dielectric the total charge density:

$$\rho = \rho_b + \rho_f$$

(Here we only consider volume bound charge) we omit σ_b

Gauss law $\Rightarrow$ $\epsilon_0 \vec{\nabla}\cdot\vec{E} = \rho = \rho_b + \rho_f = -\vec{\nabla}\cdot\vec{P} + \rho_f$

$\vec{E}$ is the total $\vec{E}$-field not just that portion generated by polarization.

If we combine two divergence terms:

$$\epsilon_0 \vec{\nabla}\cdot\vec{E} + \vec{\nabla}\cdot\vec{P} = \rho_f \rightarrow \boxed{\vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho_f}$$

$$\boxed{\epsilon_0\vec{E} + \vec{P} \equiv \vec{D}}$$ $\rightarrow$ Electric displacement.

$\Rightarrow$ Gauss law reads $\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f}$

Or, in integral form: $\boxed{\oint \vec{D}\cdot d\vec{a} = Q_{f\,enc}}$ $\rightarrow$ total free charge enclosed in the volume.

★ This is a useful way to describe Gauss Law in the context of dielectrics, because it makes reference to only free charges; we have control on free charge, bound charge appears on the way
$\rightarrow$ when we put free charge in place a certain polarization automatically ensues. and this polarization produces the bound charge. In a typical problem we know ρ_f, but we don't know ρ_b. Most of the time we can calculate $\vec{D}$ by standard Gauss Law methods.

23 March

★ We didn't use σ_b while deriving $\vec{\nabla}\cdot\vec{D} = \rho_f$ because we are ~~on~~ interested in the field inside dielectric. ~~However the~~ To do that we can enlarge the Gaussian surface until the surface of dielectric but Gauss Law still ~~state~~ give us some information about the inside the dielectric not ~~about~~ the surface. ~~outside the dielectric~~ When we want to calculate the field outside the dielectric then $\vec{P} = 0$ so, $\sigma_b = -\vec{\nabla}\cdot\vec{P} = 0$ is. Therefore, we don't need to consider σ_b outside the dielectric either.

A deceptive similarity between $\vec{E}$ and $\vec{D}$

Since $\vec{\nabla}\cdot\vec{D} = \rho_f$ just like the Gauss law and the since only difference is ρ replaced by ρ_f you may think that $\vec{D}$ is just like $\vec{E}$. But this isn't true: There is no ~~Gauss~~ Coulomb law for $\vec{D}$

$$\vec{D}(\vec{r}) \neq \frac{1}{4\pi}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho_f(\vec{r}\,')\,d\tau'$$

Similarity between $\vec{E}$ and $\vec{D}$ is more subtle.

* Divergence is not enough to determine a vector field. e.g. $\vec{E}$

  $\vec{\nabla}\cdot\vec{F} = \vec{G}$ → Derivative it is ~~true~~ valid up to a constant ~~fe~~ e.g.

Also $\vec{\nabla}\times\vec{F} = \vec{H}$ → if $\vec{F}$ has some <u>divergence-free components</u> $\vec{\nabla}\cdot\vec{F} = \vec{G}$ will not contain any information about them. Even though we know both of these, we cannot still construct $\vec{F}$ cus these are invariant <u>up to a constant</u>. Therefore we need some boundary conditions.

Think of the D.E we solved in mechanics class → To construct $\vec{F}$ we need to know $\vec{\nabla}\cdot\vec{F} = \vec{G}$, $\vec{\nabla}\times\vec{F} = \vec{H}$ + BOUNDARY CONDITIONS

* Since $\vec{\nabla}\times\vec{E} = 0$ in electrostatics most of the time we forget about that fact. Even though $\vec{\nabla}\times\vec{E} = 0$, $\vec{\nabla}\times\vec{D} \neq 0$ all the time.

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times(\epsilon_0\vec{E} + \vec{P}) = \cancel{\epsilon_0\vec{\nabla}\times\vec{E}}^{\,0} + \vec{\nabla}\times\vec{P} = \vec{\nabla}\times\vec{P}$$ and there is no reason to assume that $\vec{\nabla}\times\vec{P}$ vanishes all the time.

since $\vec{\nabla}\times\vec{D} \neq 0$ → $\vec{D} \neq \vec{\nabla}f$ → There is no potential for $\vec{D}$

* You may think that ρ_f is the only source of $\vec{D}$. Pas in $\vec{\nabla}\cdot\vec{E} = \rho$ But this isn't true in general.

$$\boxed{\text{HW) Solve Problem 4.11. After that assume that } \rho_f \text{ is the only source for } \vec{D}\text{. what happens to } \vec{E} \text{ inside and outside? Is that reasonable?}}$$

Boundary Conditions,

Boundary conditions can be restated in terms of $\vec{D}$, similar to $\vec{E}$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

BACK

Susceptibility, Permittivity, Dielectric Constant

we can write $\vec{P}$ in terms of total electric field in a linear way!

So far we have only talked about the effects of the polarization we didn't comment on the cause of $\vec{P}$. But we know that $\vec{P}$ in a dielectric results from an electric field which lines up the atomic an molecule dipoles. For many materials $\vec{P} \propto \vec{E}$ provided that $\vec{E}$ is not too strong: (can you guess why?)

χ_e: A parameter describing how polarizable the material can be

$$\boxed{\vec{P} = \epsilon_0 \underbrace{\chi_e}_{\text{electric susceptibility (dimensionless number)}} \vec{E}}$$

$\chi_e \in [0, \infty)$

* Value of χ_e depends on the microscopic structure of the ~~mate~~ dielectric. The materials obey $\vec{P} = \epsilon_0 \chi_e \vec{E}$ is called "linear dielectrics" (why??)

* $\vec{E}$ is the total field: Free charges + polarization

  e.g. if we know ext. field $\vec{E}_0$ we cannot compute $\vec{P}$ directly from $\vec{P} = \epsilon_0 \chi_e \vec{E}$

  $\vec{E}_0$ will polarize the material and then this polarization will produce its own field which then contributes to the total field and this in turn modifies the polarization ... it isn't always easy to find $\vec{E}$;

  → Simplest approach is to find $\vec{D}$ with conventional methods from ρ_f.

* In linear medias

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E} = \epsilon_0 (1 + \chi_e) \vec{E}$$

so $\vec{D} \propto \vec{E} \longrightarrow \vec{D} = \epsilon \vec{E}$ where $\boxed{\epsilon = \epsilon_0 (1 + \chi_e)}$: permittivity of the material

* In vacuum, there is no matter to polarize $\chi_e = 0$ $\epsilon = \epsilon_0$ That's why ϵ_0 is called the permittivity of free (empty) space but this description is a bit deceptive since it suggests that vacuum is a special kind of linear dielectric.

$$\boxed{\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_e}$$ relative permittivity / dielectric constant

* ϵ and ϵ_r do not contain any new information that was not available in χ_e. χ_e contains all info for linear dielectrics.

Material	ϵ_r
Vacuum	1
Helium	1.000065
Diamond	5,7

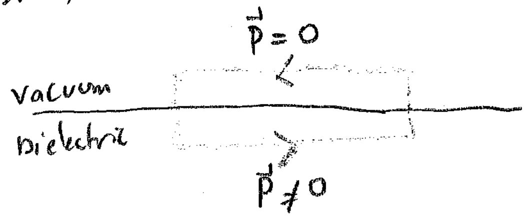
- I said that there is no reason to assume that $\vec{\nabla}\times\vec{P}=0$. Is this fact also applicable to linear dielectrics?

Because in linear dielectrics we assume that

$$\left.\begin{aligned}\vec{P}&=\epsilon_0\chi_e\vec{E}\\ \vec{D}&=\epsilon\vec{E}\end{aligned}\right\}$$

$\vec{P}$ and $\vec{D}$ are now proportional to $\vec{E}$ so should $\vec{\nabla}\times\vec{P}=0$ vanish as $\vec{\nabla}\times\vec{E}$ does?

- The short answer is no.



$\oint \vec{P}\cdot d\vec{l}\neq 0$ and by Stokes theorem

$$\int \vec{\nabla}\times\vec{P}\cdot d\vec{a}=\underbrace{\oint\vec{P}\cdot d\vec{l}}_{\neq 0} \qquad \vec{\nabla}\times\vec{P}\neq 0$$

- There is an exception: {The whole space is filled by the dielectric material. For this special case we can safely write

{Or, equivalently, if we ~~effectively~~ only consider the part of the space filled with the dielectric}

$$\vec{\nabla}\cdot\vec{D}=\rho_f \quad\text{and}\quad \vec{\nabla}\times\vec{D}=0$$

$\vec{D}$ can be calculated from ρ_f just as though dielectric weren't there.

If ρ_f ~~wasn't inside the dielectric~~ was in the empty space:

$$\vec{\nabla}\cdot\vec{E}=\rho_f \xrightarrow[\text{obtain}]{\text{to}} \vec{E}_{vac} \qquad \vec{D}=\epsilon_0\vec{E}_{vac}+\overset{0}{\vec{P}} \Rightarrow \vec{D}=\epsilon_0\vec{E}_{vac}$$

The relationship btw $\vec{D}$ and $\vec{E}_{vac}$ would be up to a constant ϵ_0.

Inside a dielectric

$$\vec{\nabla}\cdot\vec{D}=\rho_f \longrightarrow \vec{D}=\epsilon\vec{E}$$

$\vec{E}$-field inside the dielectric.

$$\epsilon\vec{E}=\epsilon_0\vec{E}_{vac}$$

$$\cancel{\epsilon_0}\epsilon_r\vec{E}=\cancel{\epsilon_0}\vec{E}_{vac}$$

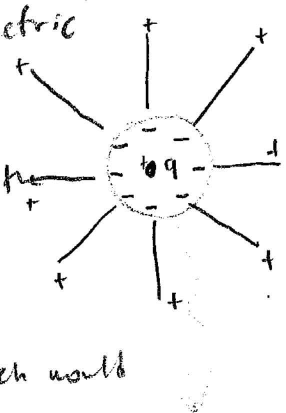
$$\boxed{\vec{E}=\frac{1}{\epsilon_r}\vec{E}_{vac}}$$

☆ when all space is filled with a homogeneous linear dielectric, the field everywhere is simply reduced by a factor of $1/\epsilon_r$

- For example: if a free charge q is embedded in a large dielectric

$$\vec{E}=\frac{1}{\epsilon_r}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}=\frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{r}$$

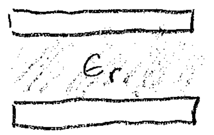
ϵ permitivity of the material.



☆ ~~This is quite~~ This result isn't surprising since ~~we knew that the polarization of the dielectric would reduce the $\vec{E}$-field~~ Introduction of the dielectric to the system would reduce the $\vec{E}$-field, which would be created as if there were no dielectric in the environment

Polarization of the medium partially shields the charge $+q$

• If a parallel-plate capacitor filled with an insulating material of dielectric constant ϵ_r what happens to its capacitance?



$\vec{E}$ will be reduced here because of the existence of the dielectric material and thence the potential by a factor of $1/\epsilon_r$

$C = \frac{Q}{V}$ will be increased by a factor of dielectric constant

$$C = \epsilon_r C_{vac}.$$

★ Placing a dielectric material in between the plates of a parallel-plate capacitor increases its capacitance. It is possible to store more energy in it!

Energy in Dielectric Systems

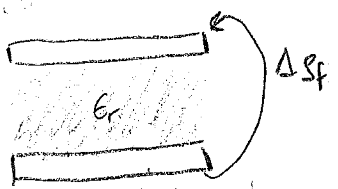
$W = \frac{1}{2} C V^2$: the work required to charge up a capacitor.

If the capacitor filled with linear dielectric, its capacitance exceeds the vacuum value by a factor of ϵ_r

$C = \epsilon_r C_{vac}$ $\rightarrow$ we need to do more work to charge it up to the same potential compared to vacuum.

• we derived the formula $W = \frac{\epsilon_0}{2} \int E^2 d\tau$ for the energy stored any electrostatic system. How does this Eq. look like in case of a dielectric?

• Assume that



• we bring $\Delta\rho_f$ at a time.
• ρ_f increases by an amount of $\Delta\rho_f$.
• Polarization will change and also ρ_b will be changed with it.
• we are only interested dq

$$\Delta W = \int (\Delta\rho_f) V d\tau$$

$\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow \Delta\rho_f = \vec{\nabla}\cdot(\Delta\vec{D}) \Rightarrow \Delta W = \int [\vec{\nabla}\cdot(\Delta\vec{D})] V d\tau$

• Consider $\vec{\nabla}\cdot[(\Delta\vec{D})V] = [\vec{\nabla}\cdot(\Delta\vec{D})]V + \Delta\vec{D}\cdot(\vec{\nabla}V)$

$$\Rightarrow \Delta W = \int \vec{\nabla}\cdot[(\Delta\vec{D})V] d\tau - \int \Delta\vec{D}\cdot(\underbrace{\vec{\nabla}V}_{-\vec{E}}) d\tau$$

$$= \underbrace{\int \vec{\nabla}\cdot[(\Delta\vec{D})V] d\tau}_{\oint[(\Delta\vec{D})V]\cdot d\vec{a} = 0} + \int \Delta\vec{D}\cdot\vec{E}\, d\tau$$

$\oint[(\Delta\vec{D})V]\cdot d\vec{a} = 0$ if we integrate over all space (Since $V \to 0$ as $r \to \infty$)

Div. Theorem
$\int \vec{\nabla}\cdot\vec{A}\, d\tau = \oint \vec{A}\cdot d\vec{a}$

$\Delta W = \int (\Delta \vec{D}) \cdot \vec{E}\, d\tau$ So far, this result applies to any material.

If we consider a linear dielectric $\vec{D} = \epsilon \vec{E}$

$$(\Delta \vec{D}) \cdot \vec{E} = \Delta(\epsilon \vec{E}) \cdot \vec{E} \Rightarrow \frac{1}{2} \epsilon\, \Delta (\vec{E} \cdot \vec{E}) = \frac{1}{2} \epsilon\, \Delta E^2 = \frac{1}{2} \Delta (\epsilon E^2) \qquad \vec{D} \cdot \vec{E} = \epsilon E^2$$

consider $\boxed{\Delta(\vec{E} \cdot \vec{E}) = 2 \Delta \vec{E} \cdot \vec{E} \Rightarrow \Delta \vec{E} \cdot \vec{E} = \frac{1}{2} \Delta (\vec{E} \cdot \vec{E})}$ $\qquad (\Delta \vec{D}) \cdot \vec{E} = \frac{1}{2} \Delta (\vec{D} \cdot \vec{E})$

$$\Rightarrow \Delta W = \int \frac{1}{2} \Delta (\vec{D} \cdot \vec{E})\, d\tau \Rightarrow \Delta W = \Delta \cdot \left(\frac{1}{2} \int \vec{D} \cdot \vec{E}\, d\tau \right) \Rightarrow \boxed{W = \frac{1}{2} \int \vec{D} \cdot \vec{E}\, d\tau}$$

Difference Between $W = \frac{1}{2} \int \vec{D} \cdot \vec{E}\, d\tau$ and $W = \frac{1}{2} \int E^2\, d\tau$

These equations consider 2 different situations.

<u>Question:</u> what do we mean by "energy of a system"?

<u>Answer:</u> The work required to ensemble the system.

- If we are interested in the work done to bring all the free and bound charges together one by one then we can use $W = \frac{1}{2} \int E^2\, d\tau$. However, notice that this will not include the work involved in stretching and twisting the dielectric molecules ([illegible]

  Think that $\oplus$ and $\ominus$ bound charges are being hold together with some springs

  $W = \frac{1}{2} \int E^2\, d\tau$ does not involve the energy stored in the spring "$\frac{1}{2} k x^2$".

- Starting with an unpolarized dielectric, if we bring the free charges one by one allowing dielectric to respond then we need to consider $W = \frac{1}{2} \int \vec{D} \cdot \vec{E}\, d\tau$. In this case, "spring energy" is also included because while deriving $W = \frac{1}{2} \int \vec{D} \cdot \vec{E}\, d\tau$ we considered how $\vec{D}$ changes (equivalently, how bound charge distribution changes, because changes in $\vec{D}$ reflects this fact), <u>In other words</u>, in this method the total energy of the system consists of 3 parts

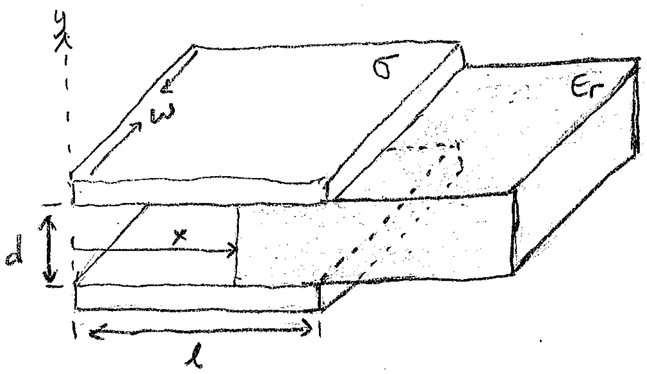
$$W_{tot} = \underbrace{W_{free}}_{\text{Electrostatic energy of free charge}} + \underbrace{W_{bound}}_{\text{electrostatic energy of bound charge}} + \underbrace{W_{spring}}_{\text{"spring" energy}}$$

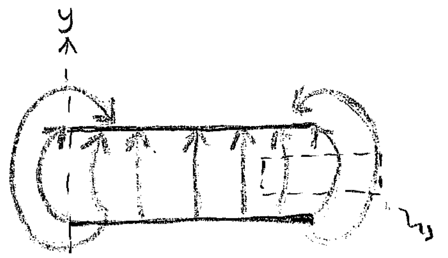
An important point here is $W_{bound} = -W_{spring}$, which means that the work we need to do to bring bound charges together is equal and opposite to the ~~work~~ energy stored in the "springs" Therefore $W = \frac{1}{2} \int \vec{D} \cdot \vec{E}\, d\tau = W_{tot} = W_{free}$.

However, if we calculate $W = \dfrac{1}{2} \int E^2\, d\tau$ we find $W_{tot} = W_{free} + W_{bound}$ but we don't take care of W_{spring}.

☆ $\oplus$ $\ominus$ If you bring this charge from inf. you do a negative work But in case of dielectrics you actually do a positive work to stretch the spring or to polarize the atom.

Forces on Dielectrics:





fringing field $\qquad \oint \vec{E}\cdot d\vec{l} = 0$

- The dielectric is pulled inside because of the fringing field.
- If $\vec{E}$ ended right at the edge of the capacitor, then $\oint \vec{E}\cdot d\vec{l}$ would not be zero.

- Even though it is extremely difficult to calculate the fringing field and calculate the force acting on the dielectric.
- However, we can come up with another method to calculate the force without calculating the fringing field.
- If we pull the dielectric out an infinitesimal distance dx, the work we do is

$$dW = F_{me}\,dx$$

- F_{me} is the force that we apply to fight with the electric force F : $F_{me} = -F$.

$$F = -\frac{dW}{dx} \text{ : electric force on the dielectric.}$$

- Energy stored in the capacitor

$$W = \frac{1}{2} C V^2$$

- Let us assume that total charge on the plates is held constant:

$$Q = CV \longrightarrow V = Q/C \text{ : } C \text{ changes as we move the dielectric}$$

$$W = \frac{1}{2} C \frac{Q^2}{C^2} = \frac{1}{2}\frac{Q^2}{C} \rightsquigarrow \text{depends on } x$$

How?

How to calculate the capacitance C?

Empty part:

$$E_e = \frac{\sigma}{\epsilon_0}, \quad V_e = \frac{\sigma d}{\epsilon_0}, \quad C_e = \frac{\cancel{\sigma} w x}{\frac{\cancel{\sigma} d}{\epsilon_0}} = \frac{w x \epsilon_0}{d}$$

Filled part:

$$E_f = \frac{\sigma}{\epsilon_r \epsilon_0}, \quad V_f = \frac{\sigma d}{\epsilon_r \epsilon_0}, \quad C_f = \frac{\cancel{\sigma}\, w(l-x)}{\frac{\cancel{\sigma} d}{\epsilon_r \epsilon_0}} = \frac{w(l-x)\epsilon_r\epsilon_0}{d}$$

$$C_{tot} = C_e + C_f = \frac{w x \epsilon_0}{d} + \frac{w l \epsilon_r \epsilon_0}{d} - \frac{w x \epsilon_r \epsilon_0}{d}$$

$$= \frac{\epsilon_0 w}{d}(x + l\epsilon_r - x\epsilon_r) = \frac{\epsilon_0 w}{d}\left[x + l\epsilon_r - (1+\chi_e)x\right]$$

$$= \frac{\epsilon_0 w}{d}\left[\cancel{x} + l\epsilon_r - \cancel{x} - \chi_e x\right]$$

$$\epsilon = \epsilon_0(1+\chi_e)$$

$$\frac{\epsilon}{\epsilon_0} = \epsilon_r = (1+\chi_e)$$

$$C_{tot} = \frac{\epsilon_0 w}{d}\left[l\epsilon_r - \chi_e x\right]$$

So,

$$F = -\frac{dW}{dx} = +\frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx}$$

$$= \frac{1}{2} V^2 \frac{dC}{dx} = -\frac{1}{2} V^2 \frac{\epsilon_0 w}{d}\chi_e$$

$$F = \underbrace{(-)}_{\text{It indicates that the force is negative in x-direction; the dielectric is pulled into the capacitor}} \frac{\epsilon_0 \chi_e w}{2d} V^2$$