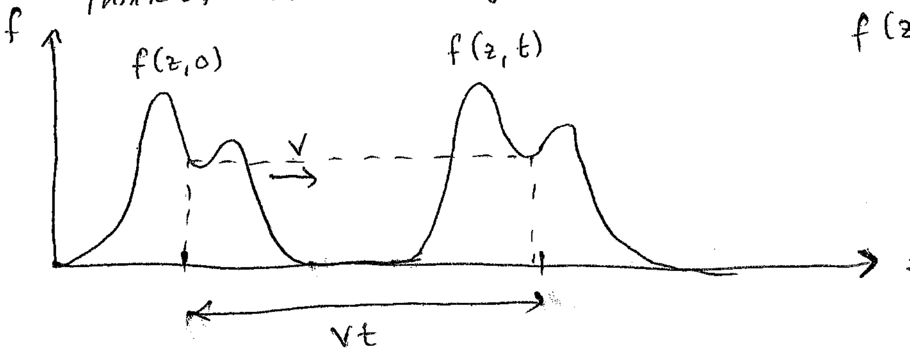
~~Electro~~

ELECTRONAGNETIC WAVES

The wave equation

A wave is a disturbance of a continuous medium that propagates with a fixed shape at a constant velocity

- How can we represent this mathematically?
  Think of a stretched string:



$f(z,t)$: represents the displacement from its equilibrium point of a string at point z at time t

- If we are given the initial shape of the string $g(z) \equiv f(z,0)$ what is the subsequent form $f(z,t)$?
- The displacement of a certain point z at $t=0$ must be the same as the displacement at $z+vt$ after time t:

$$f(z,t) = f(z-vt,0) = g(z-vt)$$: Periodicity

$z - vt$ have to be constant for the amplitude function f

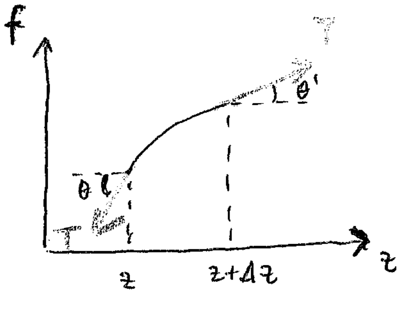
If we increase z and t at the same time z - vt should stay constant. It represents the propagation of that particular point we chose in space

For example:

$$\left.\begin{aligned} f_1(z,t) &= A e^{-b(z-vt)^2} \\ f_2(z,t) &= A \sin[b(z-vt)] \end{aligned}\right\} \text{represent waves}$$

$$\left.\begin{aligned} f_3(z,t) &= A e^{-(bz^2+vt)} \\ f_4(z,t) &= A\sin(bz)\cos(bvt)^3 \end{aligned}\right\} \text{do not represent waves.}$$

Why does a stretched string support wave motion?
$\rightarrow$ It follows from Newton's 2nd Law!



Net transverse force on the segment

$$\Delta F = T\sin\theta' - T\sin\theta$$

$\sin\theta = a/b$
$\tan\theta = a/c$
$b \approx c$ if $\theta \ll 1$

We assume that distortion of the string is not too great (T stays same and angles are small, θ, θ')
we can replace $\sin\theta$ with $\tan\theta$

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z$$

• If the mass of the string per unit length is μ, Newton's second law says:

$$\Delta F = m a = \mu \Delta z \frac{\partial^2 f}{\partial t^2} = T \frac{\partial^2 f}{\partial z^2} \Delta z$$

$$\Rightarrow \mu \cancel{\Delta z} \frac{\partial^2 f}{\partial t^2} = T \frac{\partial^2 f}{\partial z^2} \cancel{\Delta z} \Rightarrow \boxed{\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T} \frac{\partial^2 f}{\partial t^2}}$$ Wave equation

• This is known as the (classical) wave equation. Let us see it admits as solutions all functions of the form

$$f(z,t) = g(z - vt)$$

(which means, all functions that depend on variables z and t in the special combination $u \equiv z - vt$. We have just seen such functions represent waves propagating in the z direction with speed v)

<u>Proof:</u> $f(z,t) = g(z - vt) \equiv g(u)$ $\quad u \equiv z - vt$

$$\frac{du}{dz} = 1 \qquad \frac{du}{dt} = -v$$

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du} \qquad \frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2} \qquad \frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial t^2} = v^2 \frac{d^2 g}{du^2} \quad \text{and} \quad \frac{\partial^2 f}{\partial z^2} = \frac{d^2 g}{du^2}$$

$$\frac{d^2 g}{du^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \quad \text{and} \quad \frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2}$$

$$v^2 = \frac{T}{\mu} \rightarrow v = \sqrt{\frac{T}{\mu}}$$ speed of propagation

$$\Rightarrow \boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}}$$

Note that $g(u)$ can be any (differentiable) function. If the disturbance propagates without changing its shape, then it satisfies the wave equation.

• Note that functions of form $g(z-vt)$ aren't the only solutions. The wave eq. involves the square of v. Therefore we can generate another class of solutions by simply changing the sign of velocity:

$$f(z,t) = h(z + vt)$$

This represents a wave propagating in the negative z-direction.

Most general solution $f(z,t) = g(z-vt) + h(z+vt)$: wave eq is linear

Sinusoidal Waves

$$f(z,t) = A\cos\left[k(z-vt)+\delta\right]$$

~~phase difference~~ phase constant

$0 \leq \delta \leq 2\pi$ It is all about when we start the time (if we have only one wave)

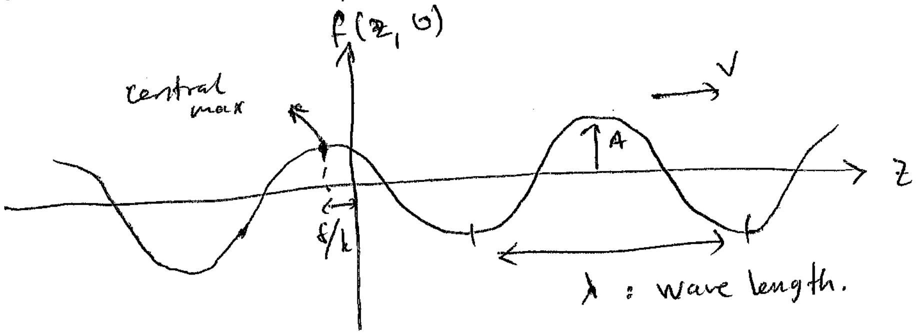
amplitude : represents the max ~~[illegible]~~ displacement from equilibrium

When $z = vt - \delta/k$ f takes it max value ; Let's call this the central max.

$kz - kvt + \delta = 0$ cos→max

$z = vt - \frac{\delta}{k}$ ✓

If $\delta = 0$ central max passes the origin at $t=0$



$k \equiv \frac{2\pi}{\lambda}$ wave number, λ → wavelength } If z advances $\frac{2\pi}{k}$ cosine completes a cycle.

For any fixed point z, the string vibrates up and down, undergoing one full cycle in a period

$$T = \frac{2\pi}{kv}$$

$$\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda} \quad : \text{frequency}$$

$\omega \equiv 2\pi\nu = kv$: angular frequency

$$f(z,t) = A\cos(kz - \overset{\omega}{kv}t + \delta) = A\cos(kz-\omega t+\delta)$$

Complex Notation : $e^{i\theta} = \cos\theta + i\sin\theta$

$$A\, e^{i(kz-\omega t+\delta)} = \cos(kz-\omega t+\delta) + i\sin(kz-\omega t+\delta)$$

$$\Rightarrow f(z,t) = \mathrm{Re}\left[A e^{i(kz-\omega t+\delta)}\right]$$

We can simply introduce a complex wave function $\tilde{f}(z,t) \equiv \tilde{A} e^{i(kz-\omega t)}$ with complex amplitude $\tilde{A} = A e^{i\delta}$

An we can write the actual wavefunction as the real part of $\tilde{f}$:

$$f(z,t) = \mathrm{Re}\left[\tilde{f}(z,t)\right]$$

Electromagnetic Waves in Vacuum

In vacuum there is no charge and current $\Rightarrow \rho = 0$ and $\vec{J} = 0$

Therefore Maxwell's Equations become

$$\vec{\nabla}\cdot\vec{E} = \cancel{\rho/\epsilon_0}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$$

$$\vec{\nabla}\times\vec{B} = \cancel{\mu_0\vec{J}} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$\longrightarrow$

$$\boxed{\begin{aligned} &(1)\ \vec{\nabla}\cdot\vec{E} = 0 \\ &(2)\ \vec{\nabla}\cdot\vec{B} = 0 \\ &(3)\ \vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t \\ &(4)\ \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \end{aligned}}$$

If we apply curl to (3) and (4):

Remember $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2 A$

$$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} \overset{\text{From (3)}}{=} \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) \overset{\text{From (4)}}{=} -\frac{\partial}{\partial t}\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{B}) - \nabla^2\vec{B} \overset{\text{From (4)}}{=} \vec{\nabla}\times\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$= \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) \overset{\text{From (3)}}{=} \mu_0\epsilon_0\frac{\partial}{\partial t}\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

So, we have

$$\underbrace{\cancel{\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{E})}}_{\text{from (1)}} - \nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \rightarrow \boxed{\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t}}$$

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{B})}_{\text{from (1)}} - \nabla^2\vec{B} = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t} \rightarrow \boxed{\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t}}$$

Therefore each component of $\vec{E}$ and $\vec{B}$ satisfies 3D wave equation

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

$\rightarrow$ same with the one we derived before only $\frac{\partial^2 f}{\partial z^2}$ is replaced with $\nabla^2 f$ it's natural generalization in 3D

Note that.

$$\frac{1}{v^2} = \mu_0 \epsilon_0 \Rightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$: permitivity of free space

$\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$: permeability of free space.

Unit analysis

$$\mu_0 \epsilon_0 \longleftrightarrow \frac{\cancel{N}}{A^2} \frac{C^2}{\cancel{N} m^2} = \frac{C^2}{A^2 m^2} = \frac{\cancel{C^2}}{\frac{\cancel{C^2}}{s^2} m^2} = \frac{s^2}{m^2}$$

$\frac{1}{\sqrt{\mu_0 \epsilon_0}} \leftrightarrow \frac{m}{s}$: speed.

$$\mu_0 \epsilon_0 = 4\pi \times 8.85 \times 10^{-12} \times 10^{-7} \frac{s^2}{m^2} \cong 108 \times 10^{-19} \cong \cancel{100}\, 10^2 \times 10^{-19} = 10^{-17}$$

$$\sqrt{\mu_0 \epsilon_0} = \sqrt{10^{-16}\, 10} = 10^{-8} \sqrt{10} \sim 10^{-8} \times 3$$

$\frac{1}{\sqrt{\mu_0 \epsilon_0}} \sim 10^8$ velocity of light $\rightarrow$ Perhaps light is also an electomagnetic wave.

Monochromatic Plane Waves

We will consider sinusoidal waves of frequency ω.

$$\vec{E}(z,t) = \tilde{E}_0 e^{i(kz - \omega t)} \qquad \vec{B}(z,t) = \tilde{\vec{B}}_0 e^{i(kz - \omega t)}$$

A wave containing only a single frequency of electromagnetic wave is called a monochromatic wave. The emagnetic spectrum contains a continuous distribution of waves.

The Electomagnetic Spectrum

Frequency (Hz)	Type
10^{21}	Gamma rays
10^{18}	x-rays
10^{16}	ultraviolet
10^{15}	visible
10^{14}	infrared
10^{10}	microwave.
10^{8}	TV, FM
10^{4}	Radio frequency

Let's plug $\tilde{\vec{E}}(z,t)$ and $\tilde{\vec{B}}(z,t)$ into Maxwell's equations to extract more information about the ~~real~~ behavior of these waves.

From ① and ② we know that $\vec{\nabla}\cdot\tilde{\vec{E}} = 0$ and $\vec{\nabla}\cdot\tilde{\vec{B}} = 0$.

If we plug the solutions in there it turns out that

$$(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$$

$\Rightarrow$ Electromagnetic waves are transverse: $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation.

Moreover, if we use $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

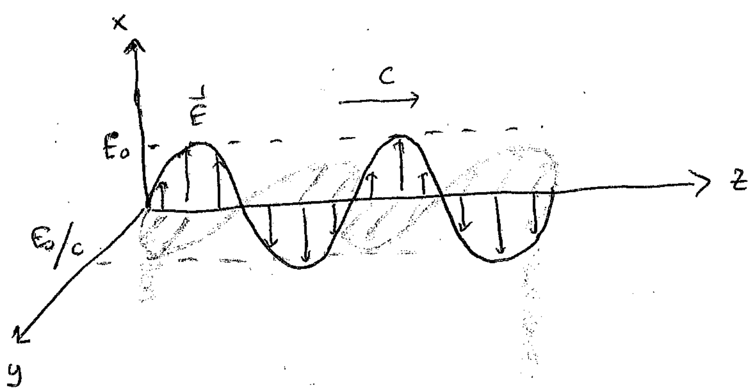
$$-k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x \quad , \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y$$

or more compactly

$$\tilde{B}_0 = \frac{k}{\omega}(\hat{z}\times\tilde{\vec{E}}_0)$$

: $\vec{E}$ and $\vec{B}$ are in phase and mutually perpendicular and their real amplitudes are

$$B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0$$



POYNTING'S THEOREM

$$W_e = \frac{\varepsilon_0}{2}\int E^2 d\tau$$

$$W_m = \frac{1}{2\mu_0}\int B^2 d\tau$$

The total energy stored in electromagnetic field:

$$U_{em} = \frac{1}{2}\int \left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) d\tau$$

- We will derive this more generally in the context of energy conservation law for electrodynamics.
- We have a charge and current distribution creating $\vec{E}$ and $\vec{B}$ at time t. In the next instant (after dt) charges move around a little bit.

Question: How much work is done by electromagnetic forces acting on these charges in the interval dt?

$\boxed{\text{WORK DONE ON CHARGE } q:}$ $\underbrace{\vec{F}\cdot d\vec{l}}_{dW} = q(\vec{E} + \vec{v}\times\vec{B})\,\vec{v}\,dt \Rightarrow \frac{dW}{dt} = q(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v} = (\vec{E}+\vec{v}\times\vec{B})\cdot(q\vec{v})$

$$= q\vec{E}\cdot\vec{v} + \underbrace{\cancel{q\,\vec{v}\times\vec{B}\cdot\vec{v}}}_{0}$$

$q = \rho\,d\tau$ and $\rho\vec{v} = \vec{J}$ $\Rightarrow$ $\frac{dW}{dt} = (\vec{E}\cancel{+\vec{v}\times\vec{B}})\cdot\underbrace{(\rho\vec{v})}_{\vec{J}}d\tau = (\vec{E}\cancel{+\vec{v}\times\vec{B}})\cdot\vec{J}\,d\tau$

$$= \vec{E}\cdot\vec{J}\,d\tau$$

For one charge q if it has velocity $\vec{v}$ the force done by $\vec{E}$ and $\vec{B}$

$\underbrace{dq(\vec{E}+\vec{v}\times\vec{B})}_{\vec{F}}\cdot\underbrace{\vec{v}\,dt}_{d\vec{l}}$ ~~But if we consider the total charge~~

$$dq = \rho\, d\tau$$

$dq\,\vec{E}\cdot\vec{v}dt + \underbrace{\cancel{dq\,\vec{v}\times\vec{B}\cdot\vec{v}}}_{0}dt$

$= dq\,\vec{E}\cdot\vec{v}dt = \rho\,d\tau\,\vec{E}\cdot\vec{v}dt = (\vec{E}\cdot\rho\vec{v})d\tau\,dt$,

And when we consider the total charge

$$dW = \left(\int_V \vec{E}\cdot\vec{J}\,d\tau\right)dt \longrightarrow \boxed{\frac{dW}{dt} = \int_V \vec{E}\cdot\vec{J}\,d\tau}$$

Ampere-Maxwell Law:

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$\vec{E}\cdot\vec{J}$: work done per unit time per unit volume.
(or power delivered per unit volume)

$$\vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_S (\vec{E}\times\vec{B})\cdot d\vec{a}}$$ Poynting's theorem.

$\frac{dW}{dt}$: the rate work is done on all the charges in a volume.

surface S encloses the volume $\mathcal{V}$

$U_{em} = \frac{1}{2}\int\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$: total energy stored in electromagnetic fields.

$\vec{S} \equiv \frac{1}{\mu_0}(\vec{E}\times\vec{B})$: energy per unit time, per unit area (transported by fields)

$$\boxed{\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oint_S \vec{S}\cdot d\vec{a}}$$

POYNTING'S THEOREM: THE WORK DONE ON THE CHARGES BY ELECTROMAGNETIC FORCE IS EQUAL TO THE DECREASE IN ENERGY STORED IN THE FIELD LESS THE ENERGY THAT FLOWED OUT THROUGH THE SURFACE.

If we rearrange the terms, maybe it will be clearer:

$$(*)\ \boxed{\frac{dW}{dt} + \oint_S \vec{S}\cdot d\vec{a} = -\frac{dU_{em}}{dt}}$$:

IN A GIVEN VOLUME V, THE ELECTROMAGNETIC ENERGY ~~ESCAPING FROM VOLUME V~~ ESCAPING, PLUS THE WORK DONE ON THE CHARGES ~~WITHIN THE SAME VOLUME~~, REDUCES THE ENERGY STORED IN THE ELECTROMAGNETIC FIELD.

- $\frac{dW}{dt}$: Work done on the charges will increase their mechanical energy (kinetic, potential, ...) Let us define a new quantity ~~the~~ called mechanical energy density u_{mech}

$$\frac{dW}{dt} = \frac{d}{dt}\int_{\mathcal{V}} u_{mech}\, d\tau$$

If we rewrite (*) : $\frac{d}{dt}\int_{\mathcal{V}} u_{mech}\, d\tau + \oint_S \vec{S}\cdot d\vec{a} = -\frac{d}{dt}\int \underbrace{\frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)}_{u_{em}} d\tau$

$$\Rightarrow \frac{d}{dt}\int_{\mathcal{V}} (u_{mech} + u_{em})\, d\tau = -\oint_S \vec{S}\cdot d\vec{a} = -\int_{\mathcal{V}} \vec{\nabla}\cdot\vec{S}\, d\tau$$

$u_{mech} + u_{em}$: total energy density.

$$\boxed{\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\vec{\nabla}\cdot\vec{S}}$$

$\updownarrow$ COMPARE

$$\boxed{\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}}$$

$\rho \longleftrightarrow u_{mech} + u_{em}$

$\vec{J} \longleftrightarrow \vec{S}$

Polarized Monochromatic Plane wave:

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)} \hat{x}$$ ~~and~~ $\Rightarrow \vec{B} = ?$

Examples will be solved: 8.1, 9.2 and details about polarization, one more question about polarization see Sakurai pg. 8

in 6.5.2025

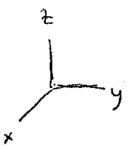
From Faraday's Law:

$$\vec\nabla \times \vec E = -\frac{\partial \vec B}{\partial t}$$

$$\tilde{\vec E}(z,t) = \tilde{\vec E}_0 e^{i(kz-wt)} \qquad \tilde{\vec E}_0 \equiv \vec E_0 e^{i\delta}$$

$$\tilde{\vec B}(z,t) = \tilde{\vec B}_0 e^{i(kz-wt)} \qquad \tilde{\vec B}_0 \equiv \vec B_0 e^{i\delta}$$

$$-\frac{\partial \tilde{\vec B}}{\partial t} = \tilde{\vec B}_0 e^{i(kz-wt)}(+w)$$



$$\vec\nabla \equiv \frac{\partial}{\partial x}\hat x + \frac{\partial}{\partial y}\hat y + \frac{\partial}{\partial z}\hat z$$

$$\tilde{\vec E}(z,t) = \vec E_0 e^{i\delta} e^{i(kz-wt)} = \left[(\tilde E_0)_x \hat x + (\tilde E_0)_y \hat y + (\tilde E_0)_z \hat z\right] e^{i(kz-wt)}$$

$$\vec\nabla \times \vec E = (\tilde E_0)_x \frac{\partial}{\partial z}\left[e^{i(kz-wt)}\right]\hat y - (\tilde E_0)_y \frac{\partial}{\partial z}\left[e^{i(kz-wt)}\right]\hat x$$

$$= (\tilde E_0)_x e^{i(kz-wt)} k\,\hat y - (\tilde E_0)_y e^{i(kz-wt)} k\,\hat x$$

$$\Rightarrow \tilde{\vec B}_0 \cancel{e^{i(kz-wt)}}\, w = k\,\cancel{e^{i(kz-wt)}}\left[(\tilde E_0)_x \hat y - (\tilde E_0)_y \hat x\right]$$

$$\left[(\tilde B_0)_x \hat x + (\tilde B_0)_y \hat y + \overset{0}{\cancel{(\tilde B_0)_z \hat z}}\right]\frac{w}{k} = (\tilde E_0)_x \hat y - (\tilde E_0)_y \hat x$$

$$\boxed{(\tilde B_0)_y \frac{w}{k} = (\tilde E_0)_x}\,(i) \quad \text{and} \quad \boxed{(\tilde B_0)_x \frac{w}{k} = -(\tilde E_0)_y}\,(ii)$$

From $\vec\nabla \cdot \vec E = \vec\nabla \cdot \vec B = 0 \longrightarrow \boxed{(\tilde B_0)_z = (\tilde E_0)_z = 0}\,(iii)$

$$\Rightarrow \boxed{\tilde{\vec B}_0 = \frac{k}{w}\left(\hat z \times \tilde{\vec E}_0\right)}\,(*)$$

- $\vec E$ and $\vec B$ have no z component and they are perpendiculer to each other. Therefore, Eqs. (i) (ii) and (iii) ~~one~~ ~~expt~~ can be summerized by a single expression (*).

Are electromagnetic waves sinusoidal in reality?

Short answer: No. They can have any shape as long as they have the form $f(z,t) = A\cos(kz - \omega t + \delta)$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad f(z,t) = Re\left[Ae^{i(kz-\omega t+\delta)}\right]$$

We can introduce a complex wavefunction:

$$\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz+\omega t)}$$ with the complex amplitude $\tilde{A} = Ae^{i\delta}$ absorbing the phase constant.

$$f(z,t) = Re\left[\tilde{f}(z,t)\right] = Re\left[Ae^{i\delta}e^{i(kz-\omega t)}\right]$$

$$= Re\left[Ae^{i(kz-\omega t+\delta)}\right] = Re\left\{A\left[\cos(kz-\omega t+\delta) + i\sin(kz-\omega t+\delta)\right]\right\}$$

$$= A\cos(kz-\omega t+\delta)$$

(Fourier Series)

* Any wave can be written as a linear combination of sinusoidal waves, and therefore if we know how sinusoidal waves behave, we know in principle how any wave behaves.

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz-\omega t)}$$

$$\tilde{\vec{E}}_0 = \vec{E}_0 e^{i\delta}$$

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\Rightarrow \vec{E}_0 = (E_0)_x\hat{x} + (E_0)_y\hat{y} + (E_0)_z\hat{z}$$

$$\tilde{\vec{E}}_0 = \underbrace{(E_0)_x e^{i\delta}}_{(\tilde{E}_0)_x}\hat{x} + \underbrace{(E_0)_y e^{i\delta}}_{(\tilde{E}_0)_y}\hat{y} + \underbrace{(E_0)_z e^{i\delta}}_{(\tilde{E}_0)_z}\hat{z}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\tilde{\vec{E}}_0 = (\tilde{E}_0)_x\hat{x} + (\tilde{E}_0)_y\hat{y} + (\tilde{E}_0)_z\hat{z}$$

$$\tilde{\vec{E}}(z,t) = (\tilde{E}_0)_x e^{i(kz-\omega t)}\hat{x} + (\tilde{E}_0)_y e^{i(kz-\omega t)}\hat{y} + (\tilde{E}_0)_z e^{i(kz-\omega t)}\hat{z}$$

$$\vec{\nabla}\cdot\tilde{\vec{E}} = \frac{\partial}{\partial z}\left[(\tilde{E}_0)_z e^{i(kz-\omega t)}\right]$$

$$= (\tilde{E}_0)_z \frac{\partial}{\partial z}\left[e^{i(kz-\omega t)}\right]$$

$$= ik(\tilde{E}_0)_z e^{i(kz-\omega t)} = 0$$

$$\left[\begin{array}{l} k \neq 0 \\ e^{i(kz-\omega t)} \neq 0 \\ \text{for all } z \text{ and } t \end{array}\right]$$

$$\Rightarrow (\tilde{E}_0)_z = 0$$

$$(E_0)_z e^{i\delta} = 0 \qquad e^{i\delta} \neq 0 \text{ anywhere}$$

$$\Rightarrow (E_0)_z = 0$$

∘ Since $\vec{\nabla}\cdot\vec{B} = 0$

$\Rightarrow (B_0)_z = 0$

Result: Since $\vec{\nabla}\cdot\vec{E} = \vec{\nabla}\cdot\vec{B} = 0$

$$(E_0)_z = (B_0)_z = 0$$

$\Rightarrow$ Electromagnetic waves are transverse: The $\vec{E}$ and $\vec{B}$ are perpendicular to the direction of propagation