Electrostatics

(can be unit of electricity)
(like mass)

Charge: It is an essential feature of matter that causes it to experience a force when placed near to another.

Mass causes an attraction between matter. Since this attraction is always attractive we simply say that there is only one "type" of matter causing this attraction.

However, electrical forces come in two variants: attractive and repulsive. Therefore, we say that charge also comes with two versions positive (+) and negative (−).

Similar charges repel each other, and opposite charges attract each other.

★ The fundamental theory of electrodynamics:

If we have a bunch of charges $q_1, q_2, \ldots$ what force do they exert on a test charge Q? How can we calculate its trajectory, while both test and source charges are in motion?

→ It is not very easy to reply to this question since the force between source and the test particle is transmitted with a finite speed; i.e. with the speed of light. So, we consider this problem step by step.

- First, we shall consider a simpler case where all source particles are stationary but test particle can move.

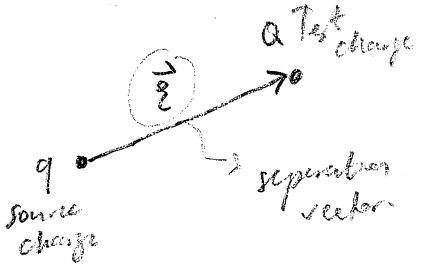
Coulomb's Law

What is the force exerted on a test charge Q due to a single point charge q which is at rest a distance ξ away?

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\xi^2}\hat{\xi}$$

ϵ_0 → permittivity of space (This is just a constant which will make our units right later)

In SI units $\epsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}$ Coulomb (C): unit of charge.



ξ : magnitude

$\hat{\xi}$: direction

The force points along the line of q and Q have the same sign (it is repulsive) and it is attractive if q or Q have opposite signs.

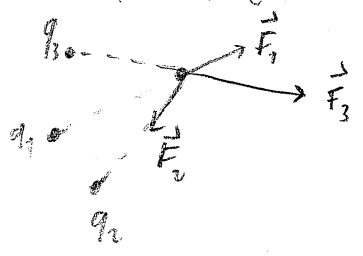
$\vec{\xi} = \vec{r} - \vec{r}'$: separation vector

$\vec{r}$ → location of Q

$\vec{r}'$ → location of q

Principle of superposition : (experimental observation)

If we have more than one source charge, the total force acting on the test charge is the summation of the individual forces from the source charges on the test charge.



$$\vec{F}_{total} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots$$

Electric Field :

If we have several charges $q_1, q_2, \dots, q_n$ at distances $\xi_1, \xi_2, \dots, \xi_n$ from q, the total force on q is

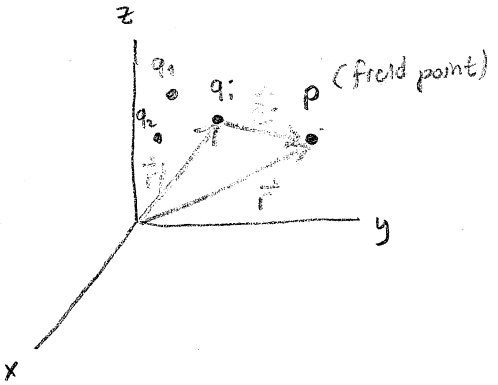
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_1 q}{\xi_1^2}\hat{\xi}_1 + \frac{q_2 q}{\xi_2^2}\hat{\xi}_2 + \dots + \frac{q_n q}{\xi_n^2}\hat{\xi}_n\right\} = \frac{Q}{4\pi\epsilon_0}\left\{\frac{q_1}{\xi_1^2}\hat{\xi}_1 + \frac{q_2}{\xi_2^2}\hat{\xi}_2 + \dots + \frac{q_n}{\xi_n^2}\hat{\xi}_n\right\}$$

$$\boxed{\vec{F} = q\vec{E}}$$

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{\xi_1^2}\hat{\xi}_1 + \frac{q_2}{\xi_2^2}\hat{\xi}_2 + \dots + \frac{q_n}{\xi_n^2}\hat{\xi}_n\right)} \quad : \text{Electric Field.}$$

in a compact form!

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{\hat{}}\frac{q_i}{\xi_i^2}\hat{\xi}_i}$$



- $\vec{E}(\vec{r})$ is the electric field of source charges.
- It is a function of position $\vec{r}$, because separation vectors depend on the position of point P.
- It varies from point to point and it depends on the configuration of point charges.
- $\vec{E}$ $[N/C]$ units.

what exactly $\vec{E}$ is?

Continuous Charge Distributions

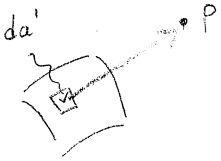
If the charge is distributed continuously over some region

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\xi^2}\hat{\xi}\, dq$$

Most of the time charge distributions are given in terms of densities.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_P \frac{\lambda(\vec{r}')}{\xi^2}\hat{\xi}\, dl' \qquad \lambda(\vec{r}') : \text{charge per unit length.}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{r}')}{\xi^2}\hat{\xi}\, da' \qquad \sigma(\vec{r}') : \text{charge per unit area.}$$



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r}')}{\xi^2}\hat{\xi}\, d\tau' \qquad \rho(\vec{r}') : \text{charge per unit volume.}$$

Field Lines

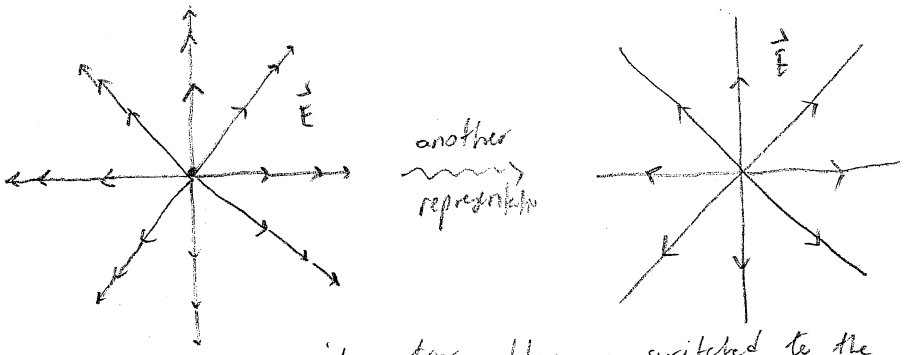
We now know how to calculate the force acting on a test charge because of a charge distribution.

CALCULATE THE ELECTRIC FIELD FORMED BY THE CHARGE DIST. $\longrightarrow$ $\vec{F} = q\vec{E}$ (test charge)

Nevertheless, now we will consider a neat representation of an electric fields via "field lines".

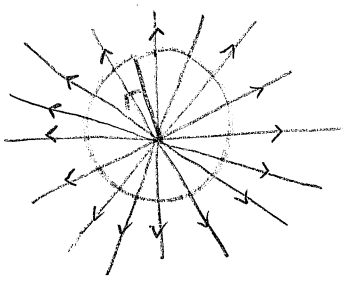
- Think of a very simple case: the electric field generated by a single source charge q situated at the origin $\vec{r}' = 0 \Rightarrow \vec{\xi} = \vec{r}$.

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$



- It may look like that we lost the strength information after we switched to the other representation. But actually we didn't. Now the strength info is contained in the density of the field lines: it is strong near the center where the field lines are close together, and weak further out, where they are relatively far apart.

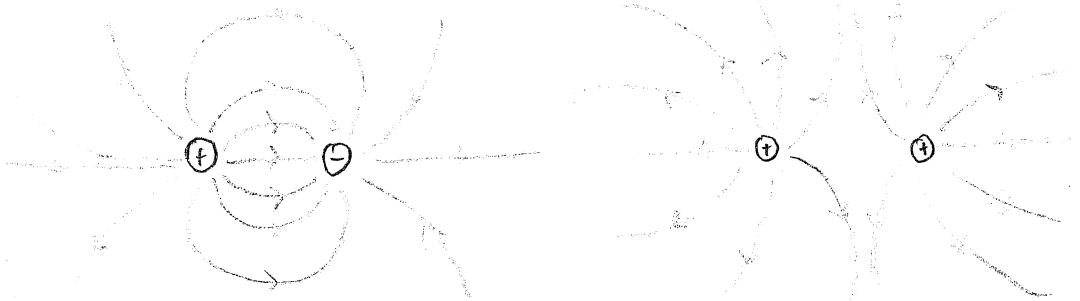
★ Field line representation in 2D is a little bit deceptive:

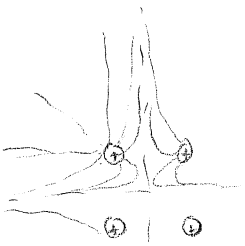


$\frac{n}{2\pi r}$: # of field lines divided by the circumference of a circle with radius r } line density goes like $1/r$, not $1/r^2$.

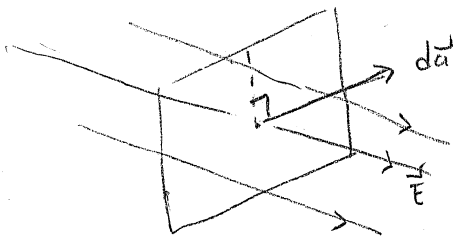
But in 3D it works well → $\frac{n}{4\pi r^2} \propto \frac{1}{r^2}$

- This representation is useful to represent more complicated fields. There are some basic rules to follow:
  - i) If we assign 8 lines for charge q, 2q deserves 16.
  - ii) Lines must be distributed in the space fairly.
  - iii) Lines begin on positive charges and end on negative ones.
  - iv) They have to stretch out to the infinity.
  - v) Field lines cannot cross.





- At this stage we will define an "electric flux" which will immitate the behaviour of the number of field lines.



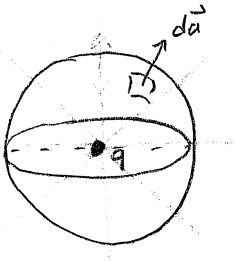
$\Phi_E \equiv \int_S \vec{E}.d\vec{a}$ $\left[\frac{N}{C}m^2\right]$: Electric field flux

$\Phi_E \propto$ # of field lines passing through the surface.

Gauss Law

- what if we calculate Φ_E over a closed surface involving a total charge of q inside?

Assume that the charge is located at the origin $\vec{r}' = 0$, $\vec{r} = \vec{\xi}$



$$\oint \vec{E}.d\vec{a} = \int \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}\right).\left(r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi = \frac{-q}{4\pi\epsilon_0}\left[\cos\theta\right]_0^{\pi} 2\pi = \frac{q}{\epsilon_0}$$

$\oint \vec{E}.d\vec{a} = \frac{q}{\epsilon_0}$: It doesn't matter which sphere we choose.

- Instead of having a single charge at the origin, let's assume that we have a bunch of scattered charges. According to principle of superposition

$$\vec{E} = \sum_{i=1}^{n} \vec{E}_i$$

- The flux through the surface which encloses all these charges

$$\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \oint \vec{E}_i \cdot d\vec{a} = \sum_{i=1}^{n} \frac{1}{\epsilon_0} q_i = \frac{Q_{enc}}{\epsilon_0}$$

$$\boxed{\oint \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}}$$ Q_{enc}: total charge enclosed within the surface.

★ Remember that the flux $\Phi_E = \oint \vec{E} \cdot d\vec{a}$ does not depend on any particular surface (r^2's cancelled during the calculation of Φ_E) It happened because of $1/r^2$ behavior of the Coulomb law.

Converting Gauss Law to a differential equation

Divergence theorem: $\int_V (\vec{\nabla} \cdot \vec{E})\, d\tau = \oint_S \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$

$$Q_{enc} = \int_V \rho\, d\tau \qquad \rho: \text{charge density.}$$

$$\int_V (\vec{\nabla} \cdot \vec{E})\, d\tau = \frac{1}{\epsilon_0} \int_V \rho\, d\tau \xrightarrow{\text{integrands must be equal.}} \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}}$$ Gauss's law in differential form

(HW): Calculate $\vec{\nabla} \cdot \vec{E}$ directly from $\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\imath}}{\imath^2} \rho(\vec{r}')\, d\tau'$

(HW) Prove explicitly. $\vec{\nabla} \times \vec{E} = 0$ by studying a point charge at the origin $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$ and considering $\int_a^b \vec{E} \cdot d\vec{\ell}$.

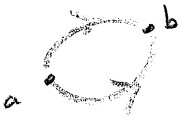
Electric Potential:

Remember that the following facts were equivalent for a curl-less vector field:

- $\vec{\nabla} \times \vec{F} = 0$
- $\int_a^b \vec{F} \cdot d\vec{\ell}$: independent of the path for a given pair of $\vec{a}$ and $\vec{b}$
- $\oint \vec{F} \cdot d\vec{\ell} = 0$ for any close loop
- $\vec{F} = -\vec{\nabla} V$

$\vec{E}$ is a curl-less vector function and above statements are applicable for it. Here we will use this feature of $\vec{E}$ and define new quantities.

- Since $\vec{\nabla}\times\vec{E}=0 \xrightarrow{\text{Stoke's Theorem}} \oint \vec{E}\cdot d\vec{\ell}=0 \longrightarrow \int_a^b \vec{E}\cdot d\vec{\ell}$: S independent



- Since we have $\int_S \vec{E}\cdot d\vec{\ell}$ = constant for any S, we will define

$$V(\vec{r}) \equiv -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{\ell}$$

$\mathcal{O}$: some standard reference we agreed beforehand
$\Rightarrow$ V only depends on $\vec{r}$.

ELECTRIC POTENTIAL

- Potential difference between two arbitrary points

$$V(\vec{b})-V(\vec{a}) = -\int_{\mathcal{O}}^{\vec{b}} \vec{E}\cdot d\vec{\ell} + \int_{\mathcal{O}}^{\vec{a}} \vec{E}\cdot d\vec{\ell} = -\int_{\mathcal{O}}^{\vec{b}} \vec{E}\cdot d\vec{\ell} - \int_{\vec{a}}^{\mathcal{O}} \vec{E}\cdot d\vec{\ell} = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{\ell}$$

$$\boxed{V(\vec{b})-V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{\ell}}$$

- The fundamental theorem for gradients states that

$$V(\vec{b})-V(\vec{a}) = \int_{\vec{a}}^{\vec{b}} (\vec{\nabla}V)\cdot d\vec{\ell} = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{\ell} \quad \Rightarrow \quad \vec{E} = -\vec{\nabla}V$$

Some comments on the potential

(unit of potential: Volt)

i) Potential $\neq$ potential energy : They are connected but they aren't the same.

ii) How to derive $\vec{E}$ out of V? V provides us a single number. But when we switch to $\vec{E}$ we have a vector quantity : 3 real components. How come this can be true?

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \hat{x}\underbrace{\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)}_{0} - \hat{y}\underbrace{\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)}_{0} + \hat{z}\underbrace{\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)}_{0} = 0$$

$\vec{E}$ is a special vector : $|\vec{E}| \propto 1/r^2$

iii) Reference point $\mathcal{O}$:

If we change our reference point from $\mathcal{O}$ to $\mathcal{O}'$

$$V'(\vec{r}) = -\int_{\mathcal{O}'}^{\vec{r}} \vec{E}\cdot d\vec{\ell} = -\int_{\mathcal{O}'}^{\mathcal{O}} \vec{E}\cdot d\vec{\ell} - \underbrace{\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{\ell}}_{V(\vec{r})} = K + V(\vec{r}) \quad \rightarrow \quad \vec{\nabla}V' = \vec{\nabla}V$$

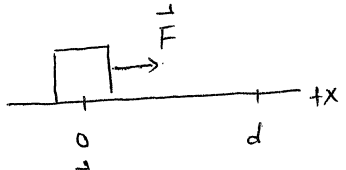
$\Rightarrow$ Both $\vec{\nabla}V'$ and $\vec{\nabla}V$ refers to the same $\vec{E}$.

Ex: Height of mount everest : above sea level or above Prague?

$\Rightarrow$ A natural spot (reference point $\mathcal{O}$) for electrostatics analogous to sea level for the altitude is a point infinitely far away from the charge

★ We will set the "zero" of potential energy at infinity.

Consider the following example:



$\vec{F} = F_0 \hat{x}$

$d\vec{s} = dx\,\hat{x}$

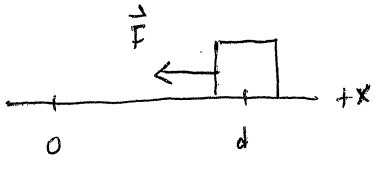
We want to calculate the work done by the constant force $\vec{F}$ ($|\vec{F}| = F_0$)

According to the directions given in the scheme:

$$W = \int_0^d \vec{F} \cdot d\vec{s} = \int_0^d (F_0 \hat{x}) \cdot (dx\,\hat{x}) = \int_0^d F_0\,dx = F_0 \int_0^d dx = F_0 d$$

→ The force $\vec{F}$ does a positive work as expected.

- Now let us consider a physically equivalent situation; the work done by the force must be the same in the following scenario:



$\vec{F} = -F_0 \hat{x}$

$d\vec{s} = +dx\,\hat{x}$ ⇝ We obtained the correct result in this case because we defined $d\vec{s}$ in this way. The boundaries of the integral automatically determine the direction of $d\vec{s}$!

$$W = \int_d^0 (-F_0 \hat{x}) \cdot (+dx\,\hat{x}) = -F_0 \int_d^0 dx = -F_0\,x\Big|_d^0 = -F_0\,(0-d) = F_0 d \checkmark$$

$$\Delta\vec{s} = \Delta x\,\hat{x} = \underbrace{(x_{final} - x_{initial})}\,\hat{x}$$

when the boundaries of the integral is chosen as $\int_d^0$, $x_{final} > x_{initial}$, because the box is moving in $-\hat{x}$ direction.

★ Therefore, if we choose $d\vec{s} = -dx\,\hat{x}$, we cannot obtain the correct result for this reason.

★ In general, once we decide on the direction of the force in question, we shouldn't worry about the sign of $d\vec{s}$. We need to choose it as $d\vec{s} = dx\,\hat{x}$ all the time because it's direction will be determined by the boundaries of the integral.

The Electric Potential

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{\ell}$$

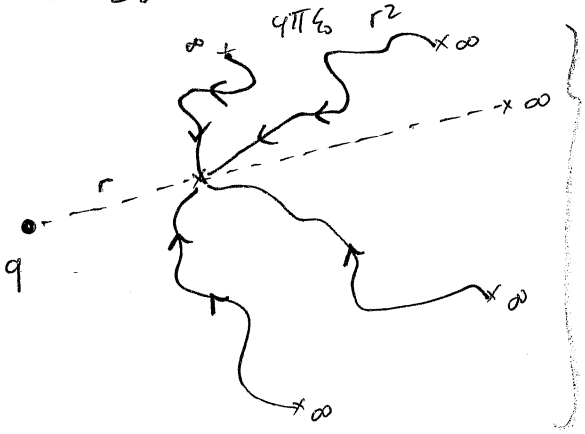
Let's calculate the potential created by a point charge.

- The electric field created by a point charge is $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$
- At this point, we should not worry about the "sign" of $\vec{E}$ because the sign of q will take care of it; if we have a positive charge $(+q)$, $\vec{E}$ will be in $\hat{r}$ direction, if the charge is negative, $\vec{E}$ will point in $-\hat{r}$ direction.
- Having determined the direction of $\vec{E}$ (encoded in q), let us determine the direction of $d\vec{\ell}$. From the previous example, we saw that $d\vec{\ell}$ must be chosen as

$$\boxed{d\vec{\ell} = dr\,\hat{r}}$$

because it is direction will be determined automatically by the boundaries of the integral.

$$\Rightarrow \vec{E}\cdot d\vec{\ell} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\underbrace{(\hat{r}\cdot\hat{r})}_{1}$$



There are infinitely many ways to get to point from $+\infty$. But since we know that

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \int_S \vec{E}\cdot d\vec{\ell} \text{ is } S \text{ independent}$$

We can choose to evaluate the integral along a straight line for simplicity.

$$\Rightarrow V(\vec{r}) = -\int_{\infty}^{r}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\Big|_{\infty}^{r} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$$

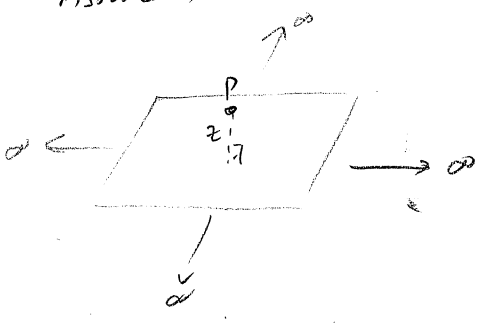
$V(\vec{r}) > 0$ if $q > 0$

$V(\vec{r}) < 0$ if $q < 0$

★ The conventional negative sign in front of the integral $V(\vec{r})$ is chosen intentionally; the region of positive charge are potential "hills", region of negative charge are potential "valleys", and the electric field points "downhill" from plus to minus.

★ Think of $\vec{E} = -\vec{\nabla}V$: Gradient of potential V → shows the steepest ascent at a given point; from minus to plus $\Rightarrow -\vec{\nabla}V$ shows the opposit direction because of $-$ sign as we expected, i.e., "downhill".

The problem with choosing $\vartheta = \infty$ arises when the charge dist. extends to ∞.
Assume that we have a uniformly charged plane: (we will study this in the examples)



If we assume $\vartheta = \infty$, what is the potential at a real height z above the plane? (we will calculate $\vec{E} = \frac{\sigma}{2\epsilon_0}$) (σ: charge density)

$$V(z) = -\int_{\infty}^{z} \frac{\sigma}{2\epsilon_0}\, dz = -\frac{\sigma}{2\epsilon_0}(z-\infty) \to \infty$$

* In fact there is no infinite charge distribution and we can safely take $\vartheta = \infty$ as our reference point. We will deal with the above problem specifically soon.

iv) Potential obey superposition principle. → total force acting on the test particle.

Coulomb superposition rule says $\vec{F} = \vec{F}_1 + \vec{F}_2 + \dots$ → ... because of the 2nd source charge

↳ force acting on the test charge because of 1st source charge.

If we divide everything by q (charge of test particle) we obtain

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots$$

$$-\vec{\nabla}V = -\vec{\nabla}V_1 - \vec{\nabla}V_2 - \dots \rightarrow \text{if we integrate this equation we obtain}$$

$$V = V_1 + V_2 + \dots \quad \text{(It is an ordinary sum, not a vector sum)}$$

v) Units of potential.

$$\vec{F}\ (N), \quad \vec{E}\ (N/C), \quad V\ [Nm/C \equiv J/C \equiv \text{Volt (V)}]$$

Poisson's Equation and Laplace's Equation

We have 2 fundamental eq. for $\vec{E}$ so far:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \vec{\nabla}\times\vec{E} = 0$$

And because of $\vec{\nabla}\times\vec{E} = 0$ we know that $\vec{E} = -\vec{\nabla}V$

How do these two equations look like in terms of V?

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot(\vec{\nabla}V) = -\nabla^2 V \Rightarrow \boxed{\nabla^2 V = -\rho/\epsilon_0}$$ Poisson's Equation.

* If there is no charge in the space $\boxed{\nabla^2 V = 0}$ Laplace's Equation.

What about $\vec{\nabla}\times\vec{E} = 0$? $\vec{\nabla}\times(-\vec{\nabla}V)$ must be zero. But we know that curl of gradient is always zero regardless what V is.

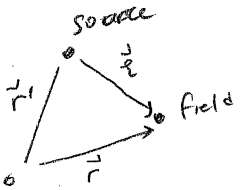
We wrote $\vec{E} = -\vec{\nabla}V$ because of $\vec{\nabla}\times\vec{E} = 0$ In return of course $\vec{E} = -\vec{\nabla}V$ will guarantee that $\vec{\nabla}\times\vec{E} = 0$. There is nothing surprising here.

To define V we only need one diff. eq: $\nabla^2 V = \rho/\epsilon_0$. But for $\vec{E}$ we need 2: $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ and $\vec{\nabla}\times\vec{E} = 0$

The Potential of a Localized Charge Distribution:

- V is defined in terms of $\vec{E}$: $V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{l}$
- If we know $\vec{E}$ there wouldn't be much point in calculating V.
- It is usually easier to first calculate V and then we can have $\vec{E} = -\vec{\nabla}V$. Because we often first know that where the charge is, i.e, ρ.
- However Poisson's equation says $\nabla^2 V = -\rho/\epsilon_0$: It gives us what ρ is when we are given V. (Take the second position derivative of V to obtain ρ)
- what we actually want is $\rho \longrightarrow V$

Discussion about $-\int_{\infty}^{r} \vec{E}\cdot d\vec{l}$ sign?



$\vec{\xi} = \vec{r} - \vec{r}'$

Potential of a point charge q located at the origin =

I use prime here to distinguish it from r.

$$V(r) = -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0}\frac{q}{r'^2}\,dr' = +\frac{q}{4\pi\epsilon_0}\frac{1}{r'}\Big|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Equipotential surfaces. It doesn't matter what θ and ϕ are.

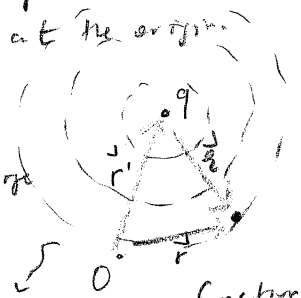
$$V(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The potential created at point r by the charge q located at the origin.

In general,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\xi}$$

ξ : distance from the charge to the point $\vec{r}$.



☆ of course, now V is a function of $\vec{r}$.

If we have a distribution of discrete charges,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\xi_i}$$

and for a continuous distribution:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\xi}\,dq$$

- For a volume charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} \frac{\rho(\vec{r}')}{\xi}\,d\tau'$$

- For a surface charge dist.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{S} \frac{\sigma(\vec{r}')}{\xi}\,da'$$

- and for a line charge dist.

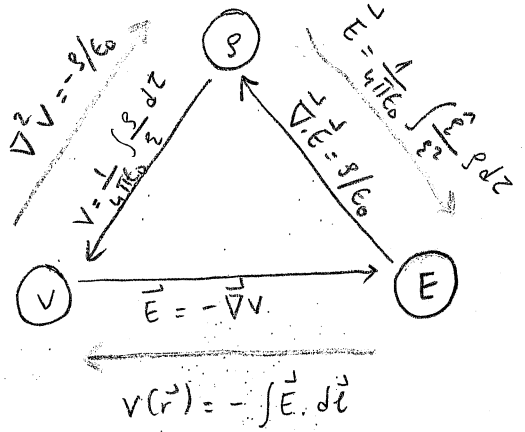
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{P} \frac{\lambda(\vec{r}')}{\xi}\,dl'$$

The solution for Poisson's equation $\nabla^2 V = -\rho/\epsilon_0$; telling us how to compute V when we are given ρ for a localized charge distribution.

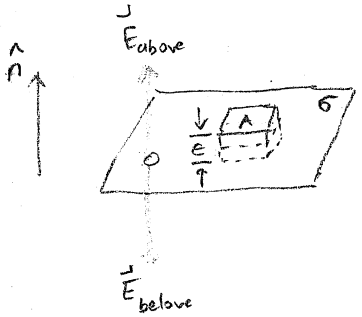
Summary:

- Generally we are given $\rho \longrightarrow \vec{E}$
- If there is a symmetry in the physical system we can use Gauss's Law
- If there is no symmetry, it is usually better to obtain $\rho \longrightarrow V$ and then it is easy to get $\vec{E}$.

THREE FUNDAMENTAL QUANTITIES FOR ELECTROSTATICS: ρ, V, $\vec{E}$



Boundary conditions for a charged surface:



$\vec{E}$ undergoes a discontinuity when you cross a surface charge σ. it doesn't matter if that surface is finite or infinite. The surface can be curved and finite, and charge distribution can be non-homogeneous.

- What we will consider is a Gaussian pillbox with infinitesimal thickness (and if σ is non-homogeneous) we will also take A infinitesimally small). This means that we are neglecting the contribution of E-flux coming from the sides of the pill box. For the top and the bottom of the pill box, the integral $\oint \vec{E} \cdot d\vec{a}$ will automatically involve the $\perp$ components of $\vec{E}$-field because of the dot product. And since we assume $\epsilon \to 0$, we ~~are~~ neglect any contribution coming from the sides of the pill box. Thus for

$$\frac{Q_{enc}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \quad \text{and} \quad \lim_{\epsilon \to 0} \oint \vec{E} \cdot d\vec{a} = E^{\perp} 2A$$

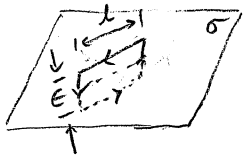
$$\Rightarrow E^{\perp} = \frac{\sigma}{2\epsilon_0} \Rightarrow E^{\perp}_{above} = \frac{\sigma}{2\epsilon_0} \qquad E^{\perp}_{below} = -\frac{\sigma}{2\epsilon_0}$$

$$\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma \hat{n}}{2\epsilon_0} - \left(-\frac{\sigma}{2\epsilon_0}\right)\hat{n} = \frac{\sigma}{\epsilon_0}\hat{n} \Rightarrow$$

The normal component of $\vec{E}$-field is discontinuous by an amount σ/ϵ_0 at the surface boundary.

- What about the tangential comp. of $\vec{E}$, i.e, $\vec{E}''$?

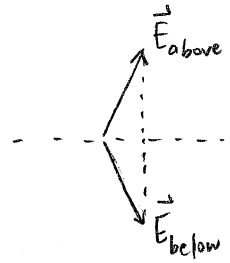
$$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \oint \vec{E} \cdot d\vec{l} = 0$$



- If we consider $\lim_{\epsilon \to 0} \oint \vec{E} \cdot d\vec{l}$ ends will not contribute. since $\epsilon \to 0$. Sides will pickup
- only the parallel component of $\vec{E}$ ($\vec{E}''$) and will yield $E''_{above}\, l - E''_{below}\, l$.

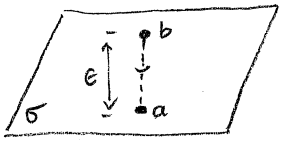
$$\Rightarrow \vec{E}''_{above}\, l - \vec{E}''_{below}\, l = 0 \Rightarrow \vec{E}''_{above} = \vec{E}''_{below}$$

Therefore, if we know that $\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$ and $E^{\parallel}_{above} = E^{\parallel}_{below}$:



$\rightsquigarrow$ surface $\Rightarrow$ $\boxed{\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}}$

what about the potential?



$$V_{above} - V_{below} = -\int_a^b \vec{E}\cdot d\vec{l}$$

$$-\lim_{\epsilon\to 0}\int_a^b \vec{E}\cdot d\vec{l} = 0 \Rightarrow V_{above} = V_{below}$$

However, the gradient of V inherits discontinuity because $\vec{E} = -\vec{\nabla}V$.

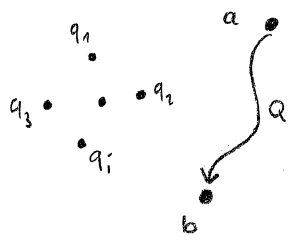
$$\vec{\nabla}V_{above} - \vec{\nabla}_{below} = -\vec{E}_{above} + \vec{E}_{below} = -\frac{\sigma}{\epsilon_0}\hat{n}$$

If we define $\frac{\partial V}{\partial n} \equiv \vec{\nabla}V\cdot\hat{n}$ (normal derivative of V)

$$\vec{\nabla}V_{above}\cdot\hat{n} - \vec{\nabla}V_{below}\cdot\hat{n} = -\frac{\sigma}{\epsilon_0} \Rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

Work and Energy in Electrostatics

The work done to move a charge



We want to move charge Q from $\vec{a}$ to $\vec{b}$ in existence of a bunch of source charges q_i. How much work we have to do?
Source charges generate $\vec{E}$ and the total force applied on charge Q will be $Q\vec{E}$. Since we want to move Q with a constant velocity we must apply a force of $-Q\vec{E}$. Therefore,

$$W = \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l} = -Q\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = Q\left[V(\vec{b}) - V(\vec{a})\right]$$

→ It is independent from the path. => Electrostatic force is conservative.

$V(\vec{b}) - V(\vec{a}) = W/Q$: Potential difference between points $\vec{a}$ and $\vec{b}$ is equal to the work per unit charge required to carry a particle from $\vec{a}$ to $\vec{b}$.

★ If we want to bring the charge Q from very far away and put it at point $\vec{r}$, the work we have to do is

$$V(\vec{r}) - \underbrace{V(\infty)}_{0} = W/Q \Rightarrow \boxed{W = Q\,V(\vec{r})}$$

=> Potential is the potential energy (the work required to set the system up) per unit charge.

The Energy of a Point Charge Distribution

How much work would it take to assemble an entire collection of point charges?
Assume that we place charges one by one: $q_i \to \vec{r}_i$

- $q_1 \to \vec{r}_1$ would require no work because there is no $\vec{E}$ yet to fight against.
- $q_2 \to \vec{r}_2$ would cost us $q_2 V_1(\vec{r}_2)$, where V_1 is the potential due to q_1.

$$\Rightarrow W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{r_{12}}\right)$$

- $q_3 \to \vec{r}_3 \Rightarrow W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$

- $q_4 \to \vec{r}_4 \Rightarrow W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right)$

$\Rightarrow$ The total work we need to do to bring 4 charges together is

$$W^{(4)} = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_1 q_4}{r_{14}} + \frac{q_2 q_3}{r_{23}} + \frac{q_2 q_4}{r_{24}} + \frac{q_3 q_4}{r_{34}}\right)$$

$\Rightarrow$ General rule: Take the product of each pair of charges and divide by their seperation distance and add it all up:

$$W = \frac{1}{4\pi\epsilon_0} \left(\frac{1}{2}\right) \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \left(\frac{q_i q_j}{r_{ij}}\right)$$

$\rightsquigarrow$ we divided it by $1/2$ because we counted each pair twice.

If we rearrange this expression as

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left[\frac{1}{4\pi\epsilon_0} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{q_j}{r_{ij}}\right]}_{V(\vec{r}_i)} \Rightarrow \boxed{W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)}$$

$V(\vec{r}_i)$: potential at $\vec{r}_i$ due to the source charges.

$\Rightarrow$ We obtained the total energy to assemble the system. Of course, it is also the energy we would get back if the system is disintegrated. This energy can also be interpreted as the energy "stored in the configuration".

The Energy of a Continuous Charge Distribution

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i) \xrightarrow{\text{continuous}} W = \frac{1}{2} \int V \rho \, d\tau \quad \text{or} \quad \overbrace{W = \frac{1}{2} \int V \lambda \, dl}^{\text{line ch. dist.}} \quad \overbrace{W = \frac{1}{2} \int V \sigma \, da}^{\text{surface ch. dist.}}$$

It is also possible to rewrite this result in terms of $\vec{E}$:

$$W = \frac{1}{2} \int V \rho \, d\tau \Rightarrow W = \frac{\epsilon_0}{2} \int (\vec{\nabla} \cdot \vec{E}) V \, d\tau$$

($\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$)

We will make use of integration by parts to re-write $W = \frac{\epsilon_0}{2}\int(\vec{\nabla}.\vec{E})\,V\,d\tau$:

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b \frac{d}{dx}(fg)\,dx = \int_a^b f\frac{dg}{dx}\,dx + \int_a^b g\frac{df}{dx}\,dx$$

$$fg\Big|_a^b = \int_a^b f\frac{dg}{dx}\,dx + \int_a^b g\frac{df}{dx}\,dx \rightarrow \boxed{\int_a^b f\frac{dg}{dx}\,dx = fg\Big|_a^b - \int_a^b g\frac{df}{dx}\,dx}$$

↑ OR

For example: To evaluate $\int_0^\infty \underbrace{x}_{u}\underbrace{e^{-x}\,dx}_{dv}$ $\rightarrow \frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$$\Rightarrow \int_0^\infty xe^{-x}dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty e^{-x}dx = 0 - e^{-x}\Big|_0^\infty = -\{0-1\} = 1$$

Similarly: $\vec{\nabla}.(f\vec{A}) = f\vec{\nabla}.\vec{A} + \vec{A}.(\vec{\nabla}f)$ — via divergence theorem →

$$\int \vec{\nabla}.(f\vec{A})\,d\tau = \int f(\vec{\nabla}.\vec{A})\,d\tau + \int \vec{A}.(\vec{\nabla}f)\,d\tau = \oint f\vec{A}\,d\vec{a}$$

$$\Rightarrow \boxed{\int_v f(\vec{\nabla}.\vec{A})\,d\tau = -\int \vec{A}.(\vec{\nabla}f)\,d\tau + \oint f\vec{A}\,d\vec{a}}$$ we will make use of this.

$$W = \frac{\epsilon_0}{2}\int(\vec{\nabla}.\vec{E})\,V\,d\tau = \frac{\epsilon_0}{2}\left[-\int_v \vec{E}.(\vec{\nabla}f)\,d\tau + \oint_s V\vec{E}.d\vec{a}\right]$$

($\vec{E}$ = $-\vec{E}$ ↑)

$$\Rightarrow \boxed{W = \frac{\epsilon_0}{2}\left(\int_v E^2\,d\tau + \oint_s V\vec{E}.d\vec{a}\right)}$$

- Consider $W = \frac{1}{2}\int V\rho\,d\tau$ again. We are integrating over a volume where the charge is located, i.e., $\rho \neq 0$. If we integrate over a larger region instead, the value of this integral would not change since in that extra volume we cover, $\rho = 0$ and it will not contribute to the total result.
- Now let us consider $W = \frac{\epsilon_0}{2}\left(\int_v E^2\,d\tau + \oint_s V.\vec{E}.d\vec{a}\right)$ As we increase the volume we consid. while evaluating the first integral, the value of $\int_v E^2 d\tau$ will increase since $E^2 > 0$. Meanwhile S will increase too but $\oint_s V.\vec{E}.d\vec{a}$ must decrease to compansate for $\int_v E^2 d\tau$'s increas.
- We know that when $r \gg 1$ $E \propto 1/r^2$, $V \propto 1/r$ and surface area $\propto r^2$ Therefore $\oint_s V.\vec{E}.d\vec{a} \propto 1/r$

★ So $\oint V.\vec{E}.d\vec{a}$ as $r \to \infty$ and $W = \frac{\epsilon_0}{2}\left(\int_v E^2 d\tau + \oint_s V.\vec{E}.d\vec{a}\right)$ will give us the correct energy.

⇒ Let's integrate over the all space and

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2\,d\tau}$$

Comments on Electrostatic Energy

(i) Inconsistency between $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r})$ (*) and $W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$ (**)

(*) can be positive or negative

(**) always positive

For ex: $+q \xrightarrow{\vec{\xi}} -q$ $\quad W = -\frac{q^2}{2}\frac{1}{4\pi\epsilon_0}\frac{1}{\xi}$

The energy to make a point charge acc. to $W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$

$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \rightarrow E^2 = \frac{1}{(4\pi\epsilon_0)^2}\frac{q^2}{r^4}$

$\Rightarrow W = \frac{\epsilon_0}{(4\pi\epsilon_0)^2}\frac{1}{2}\int \frac{q^2}{r^4} r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{dr}{r^2} = \infty$

$\Rightarrow$ so Which equation is correct? Both! But they capture the same aspect from different point of views.

- (*) does not take into account the energy to make a point charge; we simply started with one and found the work required to bring them together.
- (**) is more beneficial since it tells us the total energy stored in a charge configuration but (*) is more appropriate when we deal with point charges.
- Infinite energy for a point charge is unfortunately an embarrassment to electromagnetic theory.
- How did this inconsistency leaked into the equations regardless we derived one of them directly from the other?

HERE: $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i) \Longrightarrow W = \frac{1}{2}\int V(\vec{r})\,\rho(\vec{r})\, d\tau$

$V(\vec{r}_i)$ represents the potential due to all charges except q_i

$V(\vec{r})$ full potential.

In continuous case there is no discrimination between these two cases since $\rho(\vec{r})\,d\tau$ is vanishingly small at $\vec{r}$ and its contribution to $V(\vec{r})$ is zero.

(ii) Where is the energy stored?

(**) suggests that it is stored in the E-field. If we consider, for example, a spherical surface charge distribution. Charge is confined on the surface but E is everywhere except the surface. Where is the energy stored then? : IT DEPENDS OUR INTERPRETATION.

- In radiation theory it is useful to think that it is stored in E-field.
- In electrostatics it is stored in the region where the charge is stored.

(iii) The superposition principle : W is quadratic in E, therefore SP is not applicable for Electrostatic energy.

$$W_{tot} = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int (E_1+E_2)^2 d\tau = \frac{\epsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1\cdot\vec{E}_2)\, d\tau = W_1 + W_2 + \epsilon_0\int \vec{E}_1\cdot\vec{E}_2\, d\tau$$

$\Rightarrow$ If we double the charge everywhere, we quadruple the total energy.

Conductors:

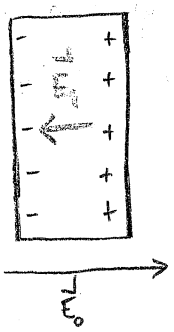
INSULATOR: Each electron is attached to a particular atom; electrons cannot move around the material. (rubber, glass,)

CONDUCTOR: One or more atoms are free to move around the material.

Some facts about conductors:

(i) Always $\vec{E}=0$ inside a conductor:

Why?



When we put a conductor inside $\vec{E}_0$, ⊕ and ⊖ charges will move in opposite directions. As they do it, they will create an opposing $\vec{E}_1$ in the opposite dir. with $\vec{E}_0$. The charges keep moving opposite direction until $\vec{E}_1$ cancels $\vec{E}_0$ inside the conductor.

ii) $\rho = 0$ inside a conductor: Gauss Law $\vec{\nabla}.\vec{E} = \rho/\epsilon_0 \rightarrow \vec{E}=0 \Rightarrow \rho=0$.

} COROLLARY. There is still charge around but exact as much plus charge as minus. $\Rightarrow$ $q_{net}=0$

iii) Any net charge stays on the surface: (Also, can be thought in a way that this is the minimal energy configuration)

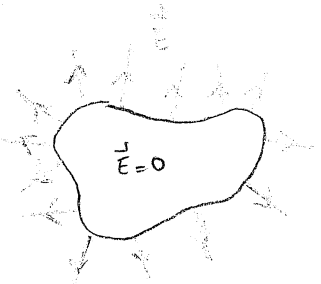
iv) A conductor is an equipotential.

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}.d\vec{\ell}$$, $\vec{a}$ and $\vec{b}$ any two points within or at the surface of a given conductor.

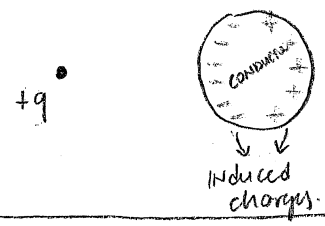
$$V(\vec{b}) = V(\vec{a})$$

v) $\vec{E}$ is perpendicular to the the surface, just outside a conductor.

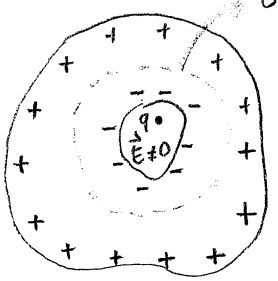
otherwise, the charge would flow around the surface and kill the tangential component



Induced Charges



If there is a cavity in the conductor



Gaussian surface.

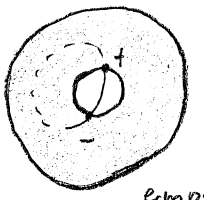
$$\oint \vec{E}.d\vec{a} = \frac{Q_{enc}}{\epsilon_0} = 0$$

$q_{inside} = q_{induced}$ on the cavity wall.

because of the surface charge of inner cavity.

- Field in the cavity is not zero.
- However cavity is completely electrically isolated from the outside world.
- Outer $\vec{E}$ will be cancelled by the charge it induces there.
- The $\vec{E}$ due to charges in the cavity will be cancelled too in the inner surface of the cavity.

Faraday Cage:



If a cavity is surrounded by a conductor has no charge inside then the field within the cavity is zero. If there was any field line going from + $\rightarrow$ − then $\oint \vec{E}.d\vec{\ell} \neq 0$ which is impossible. $\Rightarrow \vec{E}=0$ within an empty cavity. That's why we are safe in a metal car during thunderstorm.

$\rightarrow$ FARADAY CAGE.

Surface Charge and the Force on a Conductor

$\vec{E} = 0$ inside a conductor. Boundary condition requires: (all charge is accumulated on the outside of a conductor.)

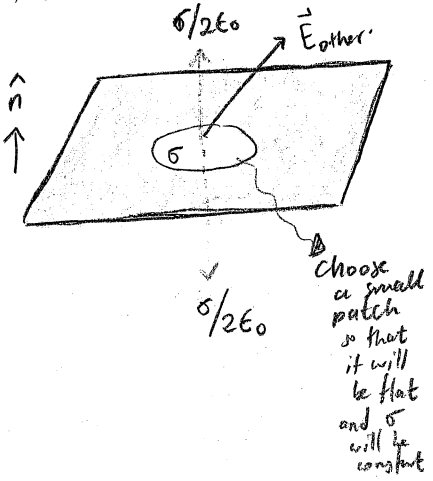
- $\vec{E}_{above} - \cancel{\vec{E}_{below}}^{\,0} = \frac{\sigma}{\epsilon_0}\hat{n}$ $\quad \vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$: immediately outside the conductor.

In presence of $\vec{E}$, the surface charge will experience a force:

$$\vec{f} = \sigma \vec{E} \quad : \text{force per unit area (pressure)}$$

But since we are considering the surface itself now, which $\vec{E}$ are we going to consider? $\vec{E}_{below}$ or $\vec{E}_{above}$? (For a conductor one of them must be zero, but the things we are going to discuss here are applicable for any surface charge)

Let's consider the following surface charge:



$\vec{E}_{other}$ is generated by the rest of the surface charge except the little patch we consider in the figure.

- Total $\vec{E}$ consists of 2 parts

$$\vec{E} = \vec{E}_{patch} + \vec{E}_{other}$$

- The patch cannot exert a force on itself. Therefore, force on the patch is due to $\vec{E}_{other}$ and this does not suffer from the discontinuity.
(Discontinuity is solely due to the charge on the patch)

$$\left.\begin{aligned} \vec{E}_{above} &= \vec{E}_{other} + \frac{\sigma}{2\epsilon_0}\hat{n} \\ \vec{E}_{below} &= \vec{E}_{other} - \frac{\sigma}{2\epsilon_0}\hat{n} \end{aligned}\right\} \quad 2\vec{E}_{other} = \vec{E}_{above} + \vec{E}_{below}$$

$$\Rightarrow \boxed{\vec{E}_{other} = \frac{1}{2}\left(\vec{E}_{above} + \vec{E}_{below}\right)} \quad : \text{average of } \vec{E}_{above} \text{ and } \vec{E}_{below}$$

* Averaging removes the contribution of the patch itself as we wanted.

* This result is applicable for any surface charge. For a conductor one of $\vec{E}_{above}$ and $\vec{E}_{below}$ is zero, depending on where the "body" of the conductor is located.

So outside of the conductor $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} \Rightarrow \vec{E}_{other} = \frac{\sigma}{2\epsilon_0}\hat{n}$

and finally $\boxed{\vec{f} = \frac{1}{2\epsilon_0}\sigma^2\hat{n}}$ outward electric pressure on the surface, tending to draw the surface into the field.

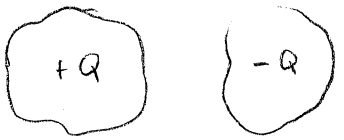
pressure or $P \equiv$

If we express the pressure in terms of the electric field outside the surface. $E = \frac{\sigma}{\epsilon_0}$

$$P = \frac{1}{2}\epsilon_0\frac{\sigma^2}{\epsilon_0^2} = \frac{1}{2}\epsilon_0 E^2 \qquad \boxed{P = \frac{\epsilon_0}{2}E^2}$$

Capacitors

If we place 2 conductors with equal and different charge next to each other the potential difference between them will be



$$V \equiv V_+ - V_- = -\int_-^+ \vec{E}\cdot d\vec{\ell}$$

We don't know how charge is distributed over two conductors.
If we try to calculate the field between them it can be very difficult because the shapes of the conductors can be complicated. Yet, what we know is $\vec{E} \propto Q$ because

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{\xi^2}\hat{\xi}\, d\tau$$ if we double q, $\vec{E}$ will be doubled

Since $\vec{E} \propto Q$ so V is because $\vec{E} = -\vec{\nabla}V$. So the proportionality between q and V is called capacitance (C)

$$\boxed{C \equiv Q/V}$$

C is purely <u>geometrical</u>: depends on the shape, size of the conductors and the separation between them

It is measured in Farads (F) : coulomb-per-volt.

*By definition q is the charge of the positive conductor and V_+ is less than that of negative one. Therefore $C > 0$. intrinsically

How to charge a capacitor?

To do that
We have to remove e^- from positive plate and carry them to the negative plate.
while doing that we do work against the $\vec{E}$ between the conducters.

→ How much work does it take to charge the capacitor up to a final amount Q?

Assume that at some point while carrying the charges the total charge accumulated on the positive plate so far is q, and hence, potential difference will be q/C. So the work we must do to transport the next piece of charge dq is

$$dW = \left(\frac{q}{C}\right)dq$$

=> The total work necessary to go from $q=0$ to $q=Q$ is

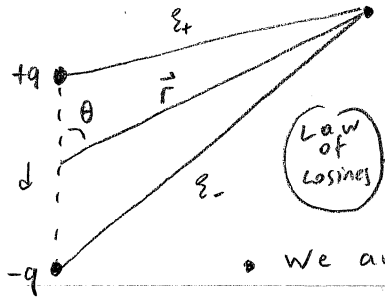
$$W = \int_0^Q \left(\frac{q}{C}\right)dq = \frac{1}{2}\frac{Q^2}{C} \quad \text{or since } Q = CV$$

$$\boxed{W = \frac{1}{2}CV^2}$$ V is the final potential of the capacitor

Multipole Expansion

If we are far away from a localized charge distribution, it looks like a point charge and the potential behaves like $\frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ where q is the total charge. What if $Q=0$? We can immediately say that potential tends to be zero too. and that is right. But we are interested here to extract a bit more information.

Consider an electric dipole:



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\xi_+} - \frac{q}{\xi_-}\right)$$

Law of cosines $\rightarrow$ $\xi_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp 2\,\frac{d}{2}\,r\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$

- We are interested in $r \gg d$ so we can neglect $d^2/4r^2$

$$\Rightarrow \xi_\pm^2 \cong r^2\left(1 \mp \frac{d}{r}\cos\theta\right)$$

Binomial expansion for $(1+x)^{-1/2} = 1 - \frac{1}{2}x + \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots$

$$\Rightarrow \frac{1}{\xi_\pm} \cong \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2}$$

neglect $(r \gg d)$

$$\frac{1}{\xi_\pm} \cong \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{d}{r}\cos\theta + \frac{1}{8}\frac{d^2}{r^2}\cos\theta \pm \cdots\right)$$

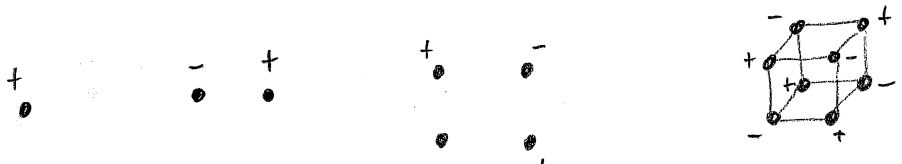
$$\Rightarrow \frac{1}{\xi_\pm} \cong \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{d}{r}\cos\theta\right)$$

Thus, $\frac{1}{\xi_+} - \frac{1}{\xi_-} = \frac{1}{r}\left\{1 + \frac{1}{2}\frac{d}{r}\cos\theta - 1 + \frac{1}{2}\frac{d}{r}\cos\theta\right\} = \frac{1}{r}\left\{\frac{d}{r}\cos\theta\right\} = \frac{d}{r^2}\cos\theta$

$$\Rightarrow \frac{1}{\xi_+} - \frac{1}{\xi_-} \cong \frac{d}{r^2}\cos\theta \Rightarrow \boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}}$$

$\rightarrow$ $V \propto 1/r^2$ at large r. It vanishes more rapidly than the potential of a point charge!

In general,



MONOPOLE $V \sim 1/r$

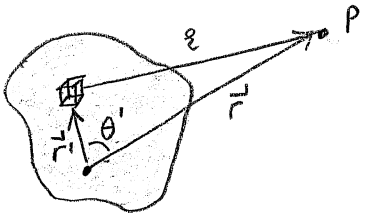
DIPOLE $V \sim 1/r^2$

QUADRUPOLE $V \sim 1/r^3$

OCTOPOLE $V \sim 1/r^4$

Now we will develop a more systematic approach to the expansion for the potential of an arbitrary localized charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\xi}\,\rho(\vec{r}')\,d\tau'$$



Law of cosines: $\xi^2 = r^2 + r'^2 - 2rr'\cos\theta'$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\theta'\right]$$

$$\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) \equiv \epsilon$$

$$\Rightarrow \xi^2 = r^2(1+\epsilon) \rightarrow \xi = r\sqrt{1+\epsilon} \rightarrow \frac{1}{\xi} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

Binomial expansion: $\frac{1}{\xi} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$

- We considered this expansion because $\epsilon \ll 1$ when $r \gg r'$, and thus, we can make approximations.
- If we write down $\frac{1}{\xi}$ in its full form (in terms of r, r' an θ')

$$\frac{1}{\xi} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \cdots\right]$$

$$-\frac{1}{2}\left(\frac{r'}{r}\right)^2 + \cos\theta'\left(\frac{r'}{r}\right) \qquad \frac{3}{8}\left(\frac{r'}{r}\right)^2\left\{\left(\frac{r'}{r}\right)^2 + 4\frac{r'}{r}\cos\theta' + 4\cos^2\theta'\right\}$$

$$\frac{3}{8}\left(\frac{r'}{r}\right)^4 - \frac{3}{2}\left(\frac{r'}{r}\right)^3\cos\theta' + \frac{3}{2}\left(\frac{r'}{r}\right)^2\cos^2\theta'$$

$$-\frac{5}{16}\left(\frac{r'}{r}\right)^3\left[\left(\frac{r'}{r}\right)^6 - 3\left(\frac{r'}{r}\right)^2 2\cos\theta' + 3\left(\frac{r'}{r}\right)4\cos^2\theta' - 8\cos^3\theta'\right]$$

$$-\frac{5}{16}\left(\frac{r'}{r}\right)^6 + \frac{15}{8}\left(\frac{r'}{r}\right)^5\cos\theta' - \frac{15}{4}\left(\frac{r'}{r}\right)^4\cos^2\theta' + \frac{5}{2}\left(\frac{r'}{r}\right)^3\cos^3\theta'$$

$$\frac{1}{\xi} = \frac{1}{r}\left\{1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\left[\frac{3\cos^2\theta' - 1}{2}\right] + \left(\frac{r'}{r}\right)^3\left[\frac{5\cos^3\theta' - 3\cos\theta'}{2}\right] + \cdots\right\}$$

The coefficients of $\frac{r'}{r}$ are Legendre polynomials $P_n(\cos\theta)$:

$$P_0(\cos\theta') = 1, \quad P_1(\cos\theta') = \cos\theta, \quad P_2(\cos\theta') = \frac{3\cos^2\theta' - 1}{2}, \quad \cdots$$

Therefore,

$$\boxed{\frac{1}{\xi} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')}$$

θ: angle between $\vec{r}$ and $\vec{r}'$

r: is a constant as far as the integration is concerned

Therefore, the potential $V(\vec{r})$ becomes:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')\,\rho(\vec{r}')\,d\tau = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\,\rho(\vec{r}')\,d\tau'$$

If we expand the summation for a couple of terms: (MULTIPOLE EXPANSION)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\underbrace{\frac{1}{r}\int \rho(\vec{r}')\,d\tau'}_{\text{MONOPOLE CONTRIBUTION}} + \underbrace{\frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}')\,d\tau'}_{\text{DIPOLE CONTRIBUTION}} + \underbrace{\frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')\,d\tau'}_{\text{QUADRUPOLE CONTRIBUTION.}} + \cdots\right]$$

★ Multipole expansion is an approximation scheme; the lowest term provides the approximate potential at large r, and the other terms improve the approximation if greater precision is required.

The Monopole and Dipole Terms:

- Monopole term: $V_{mon}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$ for $q = \int \rho(\vec{r}')\,d\tau'$

  – If we have a point charge at the origin monopole term gives us exact potential everywhere
  ⇒ All the other multipole terms vanish. ($r' = 0$)

- Dipole term: $V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}')\,d\tau'$

- If $\int \rho(\vec{r}')\,d\tau' = 0$ dipole term is dominant. Because $\int r'\cos\theta'\,\rho(\vec{r}')\,d\tau'$ term contributes more than $\int \rho(\vec{r}')\,d\tau'$.

$r'\cos\theta' = \hat{r}\cdot\vec{r}'$ where $\hat{r} = r\hat{r}$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\int \vec{r}'\,\rho(\vec{r}')\,d\tau'$$

$$\vec{p} \equiv \int \vec{r}'\,\rho(\vec{r}')\,d\tau'$$ Dipole moment of the distribution.

Therefore:

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

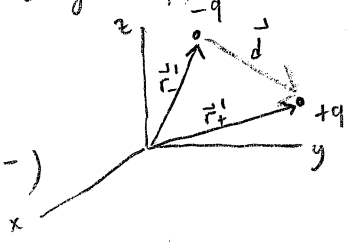
★ Dipole moment is determined by the geometry of the charge distribution.

★ Dipole moment of a collection of charges: $\vec{p} = \sum_{i=1}^{n} q_i\vec{r}_i'$

★ For the physical dipole (equal and opposite charges $\pm q$)

$$\vec{p} = q\vec{r}_+' - q\vec{r}_-' = q(\vec{r}_+' - \vec{r}_-') = q\vec{d}$$

($\vec{d}$: vector from $-$ to $+$)



• If we plug $\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i'$ into $V_{dip}(\vec{r})$: (and assuming that the origin is located at the middle of +q and -q)

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\left(q_+ \vec{r}_+ \cdot \hat{r} - q_- \vec{r}_- \cdot \hat{r}\right) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\left[\frac{d}{2}\cos\theta - \left(-\frac{d}{2}\cos\theta\right)\right] = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$

Same obtain the same result as before

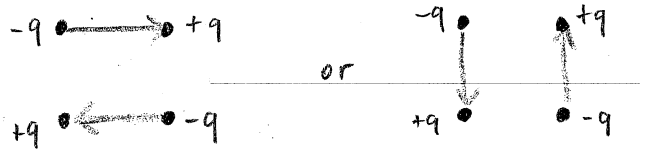
★ This is an approximate potential for physical dipole; there are higher multipole contributions.

As $r \to \infty$ V_{dip} becomes a better appr. because higher terms will die off more rapidly.

• For fixed r $d \to 0$ makes V_{dip} a better approx.

• A pure dipole can be obtained in the artificial $d \to 0$ $q \to \infty$; qd = const. limit

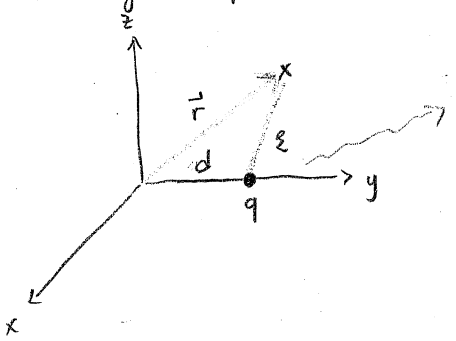
★ Dipole moments are vectors, they can be added.



A quadrupole has no net dipole moment and the potential will be dominated by the quadrupole term.

Origin of Coordinates in Multipole Expansions:

• A point charge at the origin constitutes a "pure" monopole. If it is not at the origin it is no longer a pure monopole.

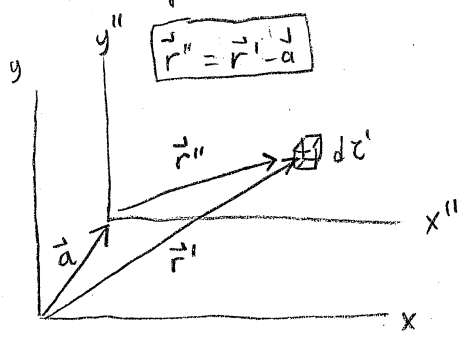


★ Therefore $\frac{1}{4\pi\epsilon_0}\frac{q}{r}$ is not quite correct for this configuration. The exact potential is given by $\frac{1}{4\pi\epsilon_0}\frac{q}{\xi}$. (when we expand $1/\xi$ we get all powers, not just the first).

=> Moving the origin can radically alter the multipole expansion.

• The "MONOPOLE MOMENT" Q does not change since total charge is independent from the coordinate system.

• Dipole moment changes as we change the origin BUT if the total charge is zero it is independent of the choice of origin.



Suppose we shifted the origin by an amount $\vec{a}$

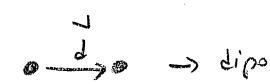
$$\vec{p}'' = \int \vec{r}'' \rho(\vec{r}')\, d\tau' = \int (\vec{r}' - \vec{a})\rho(\vec{r}')\, d\tau'$$

we still integrate w.r.t x-y-z system. However we shifted the charge dist. by $\vec{a}$ w.r.t x-y-z

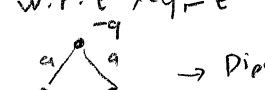
$$= \int \vec{r}' \rho(\vec{r}')\, d\tau' - \vec{a}\int \rho(\vec{r}')\, d\tau'$$

$$= \vec{p} - q\vec{a}$$

If $q = 0$ then $\vec{p} = \vec{p}''$.

=> 

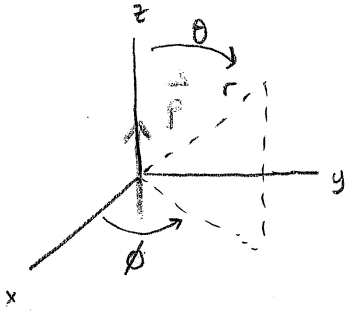
→ dipole moment $q\vec{d}$

BUT 

→ Dipole moment changes w.r.t the origin. (with respect to what origin?)

The Electric Field of a Dipole

We will calculate the E-field of a <u>pure dipole</u>:



$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

$$V_{dip}(r,\theta) = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

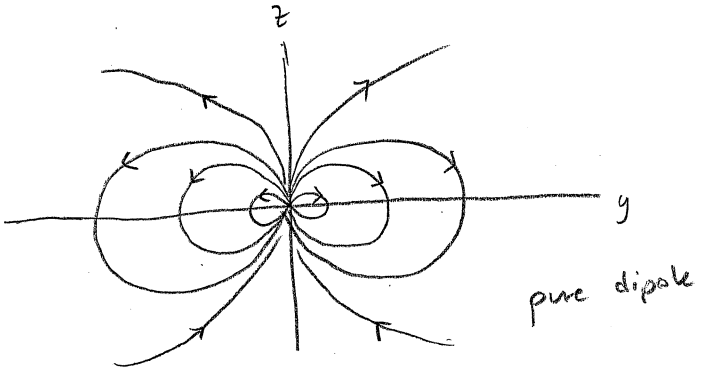
- Since $\vec{E} = -\vec{\nabla}V$ and $\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$ in spherical coordinates

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

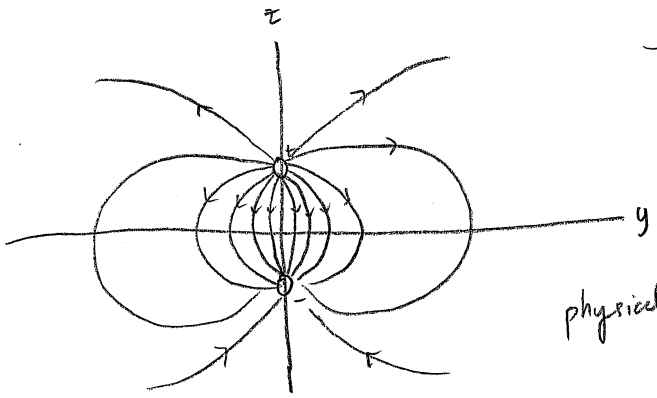
$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$



pure dipole



physical dipole