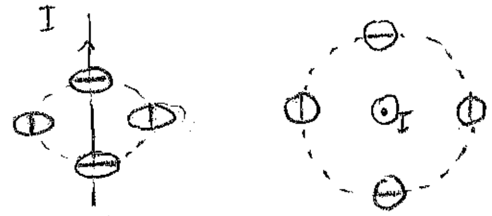
MAGNETOSTATICS (Static magnetic fields)

A charge at rest creates an electric field around it.

A moving charge creates a magnetic field around it.



Current carrying wire

From the experiment

In this section, we will be dealing with charges moving with a constant rate, and hence, creating a constant magnetic field. → They aren't accelerating on average

- How to find the direction of the magnetic field for a current carrying wire?

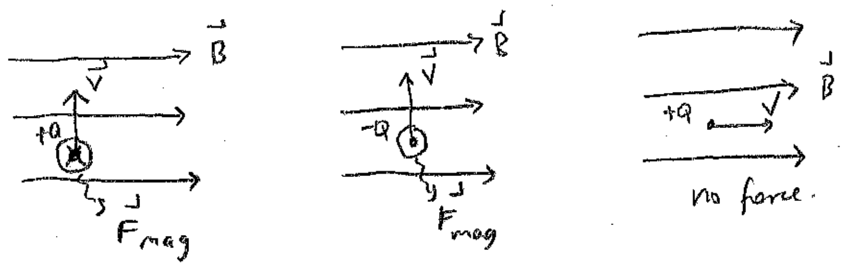
  Right-hand rule:

  Grab the wire with your right hand, your thumb must be in the direction of the current your fingers will curl around in the direction of the magnetic field.

Lorentz Law

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$$

This is a fundamental law. It has no derivation. It comes from the experiment.



- In case of both $\vec{E}$ and $\vec{B}$

$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})]$$

Magnetic forces do no work:

If Q moves an amount $d\vec{l} = \vec{v}dt$ the work done by the magnetic force is:

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}dt = 0 \quad \text{because } \vec{v} \times \vec{B} \perp \vec{v}$$

★ $\vec{F}_{mag}$ alter the direction in which particle moves but they cannot speed it up or slow it down.

Currents

★ The current in a wire is the charge per unit time passing a given point.

★ By definition opposite charges moving in opposite directions count the same.

↓

★ This definition reflects the physical fact that all physical phenomena involving moving charges depend on $q\vec{v}$; if we change the sign of q and $\vec{v}$ we will get the same answer.

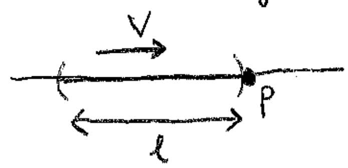
- In fact electrons move in a current-carrying wire. However, by convention the current is defined to flow in the opposite way because Benjamin Franklin assumed in the very early days of electricity that positive charges moving.

- Current is measured in Amperes(A) : $1A = 1C/s$

- If a line charge (λ) dist. travelling down a wire at speed V

$$\boxed{I = \lambda V} \; ; \; \frac{C}{m}\frac{m}{s} = \frac{C}{s}$$

When you think of a segment of this line



$\lambda \ell$: total amount of charge in this line segment.

All the charge will pass from point P in

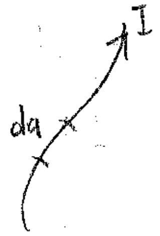
$$\Delta t = \frac{\ell}{V} \quad \Rightarrow \quad I = \frac{\lambda \ell}{\frac{\ell}{V}} = \lambda V$$

- Current is defined to be a vector:

$$\boxed{\vec{I} = \lambda \vec{V}}$$

- Note that the direction of flow is dictated by the shape of the wire we need to be careful about what we mean by $\vec{I}$. $\vec{I}$ has its meaning in wire's "curved" 1D space. We aren't interested in what kind of shape the wire has in 3D space; $\vec{I}$ simply means in any case that it flows along the cable.

- The magnetic force on a segment of current-carrying wire



$$d\vec{F}_{mag} = (\vec{V} \times \vec{B})dq$$

$$\vec{F}_{mag} = \int (\vec{V} \times \vec{B})\, dq = \int (\vec{V} \times \vec{B})\, \lambda\, d\ell = \int (\vec{I} \times \vec{B})\, d\ell$$

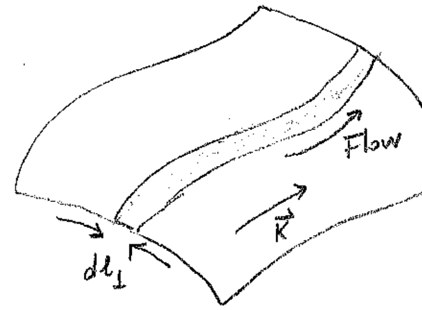
We can define that $d\ell$ and I are in the same direction and re-write above eq. as

$$\boxed{\vec{F}_{mag} = \int I \, (d\vec{\ell} \times \vec{B})}$$

Since we are considering magnetostatics, I is constant (charges move with a constant rate). Therefore:

$$\boxed{\vec{F}_{mag} = I \int d\vec{\ell} \times \vec{B}}$$

- When charge flows over a surface, we describe it by the surface current density $\vec{K}$



Consider a strip of infinitesimal width $dl_\perp$ running parallel to the current. If the current in this ribbon is $d\vec{I}$, the surface current density is

$$\boxed{\vec{K} \equiv \frac{d\vec{I}}{dl_\perp}}$$

* Current per unit width perpendicular to flow.

$$\boxed{\vec{K} = \sigma \vec{v}}$$

- $\vec{K}$ will vary from point to point over the surface according to the variations in σ and $\vec{v}$
- The magnetic force on the surface current is

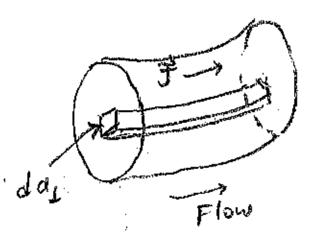
$$\boxed{\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) \sigma \, da = \int (\vec{K} \times \vec{B}) \, da}$$

* If the current flows through a volume (a three dimensional region) we describe it with a volume current density $\vec{J}$. Consider a tube of infinitesimal cross section $da_\perp$ running parallel to the flow. If the current in this tube is $d\vec{I}$ the volume current density

$$\boxed{\vec{J} = \frac{d\vec{I}}{da_\perp}}$$

The magnetic force on a volume current

$$\boxed{\vec{F}_{mag} = \int (\vec{v} \times \vec{B}) \rho \, d\tau = \int (\vec{J} \times \vec{B}) \, d\tau}$$



Continuity Equation:

$$\vec{J} = \frac{d\vec{I}}{da_\perp} \xrightarrow[\text{or surface}]{\text{current crossing}} \quad I = \int_S J \, da_\perp = \int_S \vec{J} \cdot d\vec{a}$$

Total charge per unit time leaving a closed surface S: outward flow decreases the charge left in $\mathcal{V}$.

$$\oint_S \vec{J} \cdot d\vec{a} = \underbrace{\int_{\mathcal{V}} (\vec{\nabla} \cdot \vec{J}) \, d\tau}_{\text{Div. theorem}} \quad \xrightarrow[\text{is conserved}]{\text{Charge}} \quad \int_{\mathcal{V}} (\vec{\nabla} \cdot \vec{J}) \, d\tau = -\frac{d}{dt} \int_{\mathcal{V}} \rho \, d\tau = -\int_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t} \right) d\tau$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}} \quad \text{Continuity equation}$$

The Biot-Savart law

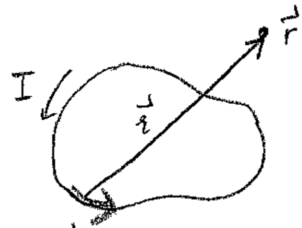
The Biot-Savart law will deal with stationary currents in magnetostatics.

As I mentioned before we will deal with

$$\vec{\nabla} \cdot \vec{J} = 0$$

Magnetic field generated by a steady field is given by the Biot-Savart law:

$$\boxed{B(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}}$$



$\mu_0 = 4\pi \times 10^{-7} \, N/A^2$: Permeability of space

Units of $\vec{B}$: Teslas $\rightarrow$ $1T = \frac{1N}{A \cdot m}$

Force per unit moving charge $\rightarrow \left(\frac{1N}{C\,V} \right)$

* Biot-Savart law plays the role of Coulomb's law in electrostatics. $1/\mathscr{r}^2$ behavior is common to both laws.

• For surface and volume currents Biot-Savart law becomes:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, da' \quad \text{and} \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

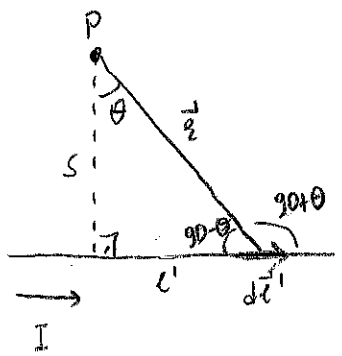
★ Is it possible to write down the corresponding formula for a moving point charge

$$B(\vec{r}) = \frac{\mu_0}{4\pi}\, \frac{q\vec{v} \times \hat{\mathscr{r}}}{\mathscr{r}^2} \rightarrow \text{THIS IS WRONG.}$$

because a point charge does not constitute a steady current; Biot-Savart law only holds for steady currents.

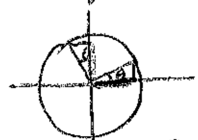
★ Superposition principle also applies to magnetic fields: If we have a collection of source currents, the net field is the vector sum of the fields due to each of them taken seperately.

<u>Example 5.5</u> : Magnetic field a distance s from a long-straight wire carrying a steady current I.



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\, I \int \frac{d\vec{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

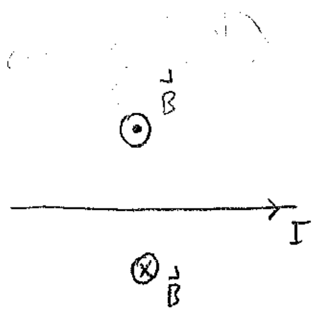
$d\vec{\ell}' \times \hat{\mathscr{r}}$: out of the page with magnitude $d\ell' \sin\left(\frac{\pi}{2}+\theta\right) = d\ell' \cos\theta$



Apparently, we can write everything (integrant) in terms of θ only.

$$B = \frac{\mu_0 I}{4\pi}\int \frac{\cos\theta}{\mathscr{r}^2}\, d\ell' \qquad \tan\theta = \frac{\ell'}{s} \Rightarrow d\ell' = \frac{s}{\cos^2\theta}\, d\theta$$

$$= \frac{\mu_0 I}{4\pi}\int \frac{\cancel{\cos\theta}}{\mathscr{r}^2}\, \frac{s}{\cos^{\cancel{2}}\theta}\, d\theta = \frac{\mu_0 I s}{4\pi}\int \frac{d\theta}{\mathscr{r}^2 \cos\theta}$$



Also, $\cos\theta = s/\mathscr{r} \Rightarrow 1/\mathscr{r}^2 = \dfrac{\cos^2\theta}{s^2}$

$$\Rightarrow B = \frac{\mu_0 I \cancel{s}}{4\pi}\int \frac{\cos^{\cancel{2}}\theta}{s^{\cancel{2}}}\, \frac{d\theta}{\cancel{\cos\theta}} = \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2} \cos\theta\, d\theta$$

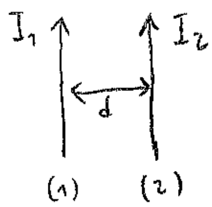
$$= \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right) \quad : \text{for any finite segment of wire}$$

★ A segment of wire by itself cannot support a steady current, but it might be a piece of some closed circuit and above expression represents its contribution to the total field.

For infinite wire: $\theta_1 = -\pi/2$ and $\theta_2 = +\pi/2$

$$\boxed{B = \frac{\mu_0 I}{2\pi s}} \qquad B \propto 1/s \text{ ; just like the E-field.}$$

Let's also calculate the force of attraction between 2 long and parallel wires a distance d apart carrying currents I_1 and I_2



The field at (2) due to (1) $\frac{\mu_0 I_1}{2\pi d}$ into the page

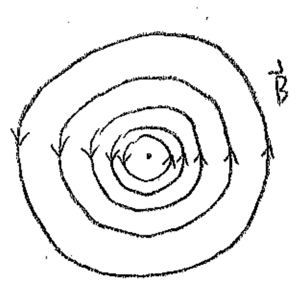
And the Lorentz force: $\vec{F}_{mag} = I \int d\vec{l} \times \vec{B}$: to the left : the force is attractive.

$$F = \frac{I_2 \mu_0 I_1}{2\pi d} \int dl$$ ⇝ it is infinite, not surprisingly.

But we can still calculate force per unit length:

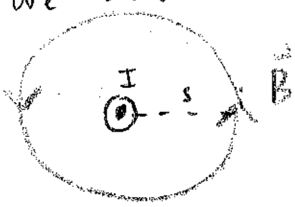
$$\frac{F}{\int dl} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{d}$$

Divergence and Curl of B for Straight-line currents



B-field generated by an infinite wire with a steady-current I coming out of the page. It definitely has a non-zero curl. So, that is what we are gonna calculate.

We know that



$$|\vec{B}| = \frac{\mu_0 I}{2\pi s} \qquad \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi s} \oint dl = \frac{\mu_0 I}{\cancel{2\pi s}} \cancel{2\pi s}$$
$$= \mu_0 I$$

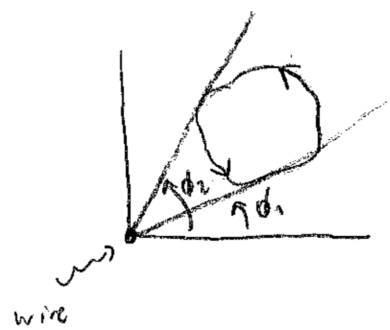
☆ The result is independent of s. because B decreases at the same rate $2\pi s$ increases

→ This result is valid for any closed paths;
In cylindrical coordinates $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$
and if current is flowing along z-axis;

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

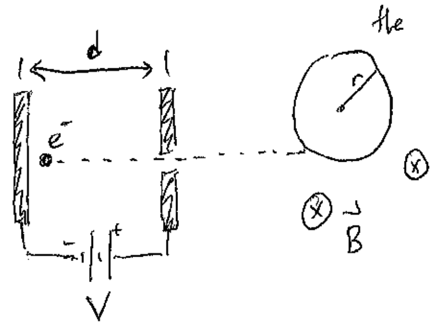
$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s} s\,d\phi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I$$

→ we assume here that the closed loop encircles the wire exactly once. if it went around twice ϕ would run from 0 to 4π.
if it didn't enclose the wire at all then ϕ would go from ϕ_1 to ϕ_2 and back again $\int d\phi = 0$



Example:

Cyclotron Motion: The electrons are accelerated in between two metal plates which are kept ~~under~~ the potential difference V. Then, they get into a region with magnetic field as given in the figure. What would be e/m in terms of B, r, V?



$$eV = \frac{1}{2} m v^2 \rightarrow \sqrt{\frac{2eV}{m}} = v$$ e^- speed after acceleration

$$F_{mag} = m\frac{v^2}{r} \Rightarrow e\cancel{v}B = m\frac{v^{\cancel{2}}}{r} \Rightarrow eB = \frac{m}{r}\sqrt{\frac{2eV}{m}}$$

$$e^2B^2 = \frac{m^2}{r^2}\,\frac{2eV}{m} \Rightarrow \boxed{\frac{e}{m} = \frac{2V}{r^2B^2}}$$

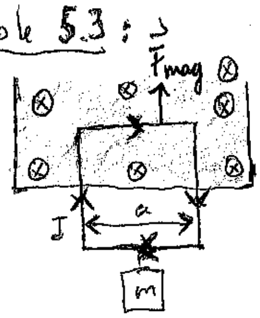
$\frac{J}{V}\,\frac{N^2}{C\,s^2}$

$\frac{Nm}{N s^2}$

$\frac{m\,C}{s^2\,m} = \frac{C}{m}$

☆ We can measure V, r, B from the experiment to learn about the fundamental e/m.

Example 5.3:



For what current in the loop, would the magnetic force upward exactly balance the gravitational force downward?

★ The current must circulate clockwise because $\vec{I}\times\vec{B}$ in the horizontal segment. The magnitude of the force is then

$$F_{mag} = IBa = mg \Rightarrow \boxed{I = \frac{mg}{Ba}}$$

Ex 5.4:

Ex 5.6:

Ex 5.7:

Ex 5.8:

Ex 5.9:

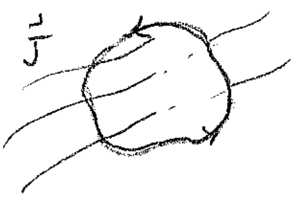
Ex 5.10:

Ex for bound charges $\xrightarrow{\text{back}}$

- Now, suppose we have a bundle of straight wires. Each wire passing through the loop contributes $\mu_0 I$, and those outside contribute nothing. The line integral will be

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \rightsquigarrow \text{total current enclosed by the integration path.}$$

- If the flow of charge is represented by a volume current density $\vec{J}$, the enclosed current is



$$I_{enc} = \int \vec{J}\cdot d\vec{a}$$ $\Rightarrow$ the integral is taken over the surface which is bounded by the loop.

- Remember, Stoke's theorem: $\int(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$. Therefore, $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$ can be written

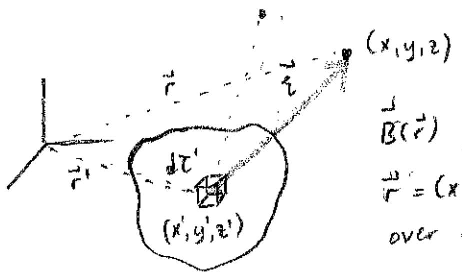
$$\int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0\int \vec{J}\cdot d\vec{a} \quad \text{and hence} \quad \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$

- This derivation is restricted to infinite straight line currents. For other cases, we have no right to assume that $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ applies to them. So, we need to derive $\vec{\nabla}\cdot\vec{B}$ and $\vec{\nabla}\times\vec{B}$ formally starting from the Biot-Savart law.

The Divergence and Curl of $\vec{B}$

For volume currents:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$



$\vec{B}(\vec{r})$ gives the magnetic field at $\vec{r} = (x,y,z)$ in terms of an integral over current distribution $\vec{J}(x',y',z')$

$\vec{B}$: function of (x,y,z)
$\vec{J}$: function of (x',y',z')
$\vec{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$
$d\tau' = dx'dy'dz'$

Integration will be taken w.r.t primed coordinates. Divergence and curl are to be taken w.r.t. unprimed coordinates.

apply $\vec{\nabla}$

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

We will make use of the product rule:
$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\,(\vec{\nabla}\times\vec{B})$$

$$\Rightarrow \vec{\nabla}\cdot\left(\frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\underbrace{(\vec{\nabla}\times\vec{J})}_{0} - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$0 \rightsquigarrow \vec{J}$ doesn't depend on unprimed coordinates

Think of $\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$. Derivative is taken w.r.t. unprimed coordinates. Since $\vec{\mathscr{r}}$ contains both primed and unprimed terms, we consider where (x',y',z') are constant and (x,y,z) are changing. Therefore, $\hat{\mathscr{r}}/\mathscr{r}^2$ will always be radially outwards for any given (x',y',z') and won't have any curl.

$\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ and $\vec{\nabla}\times\vec{J}$ are zero $\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

* The divergence of magnetic field is zero.

• Applying curl to Biot-Savart law:

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)d\tau'$$

• We will make use of another product rule:

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

$$\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \cancel{\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\vec{\nabla}\right)\vec{J}} - (\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \vec{J}\underbrace{\left[\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right]}_{4\pi\delta^3(\vec{\mathscr{r}})} - \cancel{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}(\vec{\nabla}\cdot\vec{J})}$$

• $\vec{J}$ is a function of x', y', z'. We drop the terms involving the derivatives of $\vec{J}$.

$$\Rightarrow \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')4\pi\delta^3(\vec{\mathscr{r}})\,d\tau' - \underbrace{\frac{\mu_0}{4\pi}\int(\vec{J}\cdot\vec{\nabla})\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'}$$

we showed this before.

HW Show that this term is 0.

$$\Rightarrow \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\,4\pi\,\delta^3(\vec{r}-\vec{r}')\,d\tau' = \frac{\mu_0}{4\pi}\vec{J}(\vec{r})4\pi = \mu_0\vec{J}(\vec{r}) \rightarrow \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})}$$

* Above equation is not restricted to straight-line currents.

Applications of Ampère's Law

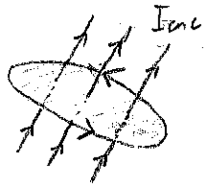
$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$ → Ampère's Law in differential form.

By using Stoke's theorem: $\int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \oint\vec{B}\cdot d\vec{l} = \mu_0\boxed{\int\vec{J}\cdot d\vec{a}}$ → Total current passing through a surface $\equiv I_{enc}$: current enclosed by the amperian loop.

$\Rightarrow \boxed{\oint\vec{B}\cdot d\vec{l} = \mu_0 I_{enc}}$ → Ampere's law in integral form.

* How to decide which direction through the surface corresponds to a positive current? And correspondingly which way around the loop are we supposed to go?

→ Right hand rule: If the fingers of our right hand indicate the direction of integration around the boundary, then our thumb defines the direction of positive current



• Electrostatics: Coulomb → Gauss Law
• Magnetostatics Biot-Savart → Ampère

* For currents with appropriate symmetry, integral form of Ampère's Law offer very efficient ways for calculating the magnetic field.
* Like Gauss Law, Ampère's Law is always true but it isn't always useful.
* Like Gauss Law, it is useful when the symmetry of the problem allows us to pull B outside the integral. If it doesn't work we need to use Biot-Savart law. Ampère's law is useful for
  i) Infinite straight lines
  ii) Infinite planes
  iii) Infinite solenoids
  iv) Toroids.

Comparison of Magnetostatics and Electrostatics

Maxwell's equations for:

ELECTROSTATICS

$\vec{\nabla}.\vec{E} = \rho/\epsilon_0$ (Gauss's Law)

$\vec{\nabla}\times\vec{E} = 0$

Once ρ is given, we can determine $\vec{E}$ with boundary condition $\vec{E}\to 0$ far from all charges.

They essentially contain the same information as Coulomb's law + principle of superposition

MAGNETOSTATICS

$\vec{\nabla}.\vec{B} = 0$

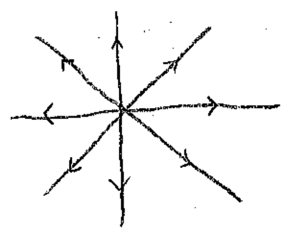
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (Ampere's Law)

Once $\vec{J}$ is given, we can determine $\vec{B}$ with boundary condition $\vec{B}\to 0$ far from all charges.

They contain same information as Biot-Savart law + Principle of superposition

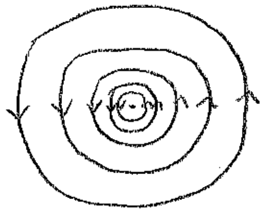
☆ Together with $\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$, these equations summarize all electrostatics and magnetostatics.

ELECTRIC FIELD



- Originate on + charges terminate on - charges.
- Diverges away from positive charge.

MAGNETIC FIELD



- B-lines do not begin or end anywhere. If they did, it would require a nonzero divergence
- They either form closed loops or extend out to infinity.

☆ There are no point sources for $\vec{B}$, but there are point sources for $\vec{E}$! $\longrightarrow$ $\vec{\nabla}.\vec{B} = 0$!

☆ Coulomb and others thought there were magnetic monopoles.

☆ It was Ampère who first speculated that all magnetic effects spring from electric charges in motion (currents).

☆ Today we know that Ampère was right and there is no concrete experiment showing the existence of magnetic monopoles.

☆ Magnetic effects take place if charges moving relative to each other.

☆ $F_E \gg F_B$: just compare μ_0 and ϵ_0. If test and source charges have to move at velocities comparable to c, then F_B approaches F_E in strength. If so how do we detect F_B?

$\mu_0 = 4\pi\times10^{-7}\,N/A^2$

$\epsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{N\cdot m^2}$

☆ Collective behavior of huge quantities of charge !!

Magnetic Vector Potential

(Remember Ph 2 for vector calculus)

$\vec{\nabla}\times\vec{E}=0 \longrightarrow \vec{E}=-\vec{\nabla}V \Rightarrow$ Similar to this $\Rightarrow$ If $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$ $\vec{A}$: vector "potential" in magnetostatics.

• If we think about the divergence and curl of $\vec{B}$ in terms of vector potential $\vec{A}$:

$\rightarrow \vec{\nabla}\cdot\vec{B}=0$ is trivial since $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})$ is always zero.

(*) $\rightarrow \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$ $\boxed{\text{Remember: } \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$

Remember that: $\vec{E}=-\vec{\nabla}V$ we can add any scalar function to V whose gradient is 0. It wouldn't change $\vec{E}$.

Similarly: we can add $\vec{A}$ and vector function whose curl is 0 and it wouldn't change $\vec{B}$.

$\Rightarrow$ we can use this feature to eliminate $\vec{\nabla}\cdot\vec{A}$ from (*) — so at this point we are using above feature

How? Assume we have an original potential $\vec{A}_0$ which isn't divergenceless. And let's add it some other vector potential which can be written as a gradient of a scalar function:

$\vec{A}=\vec{A}_0+\vec{\nabla}\lambda \rightarrow \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2\lambda$

• So, if we can find a scalar function satisfying $\boxed{\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0}$ then $\vec{\nabla}\cdot\vec{A}$ can be zero

• We also know from ~~Laplace~~ Poisson equation that $\boxed{\nabla^2 V = -\rho/\epsilon_0}$

• In above 2 equations $\vec{\nabla}\cdot\vec{A}_0$ is in place of ρ/ϵ_0 as the "source".

• We also saw before that the potential of a charge dist. $\rho(\vec{r})$ when $\rho\to 0$ at infinity is

$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{\mathscr{r}}d\tau'$ which is a solution for $\nabla^2 V = -\rho/\epsilon_0$.

• By using this fact we can immediately write down $\boxed{\lambda = \frac{\epsilon_0}{4\pi\epsilon_0}\int\frac{\vec{\nabla}\cdot\vec{A}_0}{\mathscr{r}}d\tau'}$ with the condition $\vec{\nabla}\cdot\vec{A}_0\to 0$ at infinity

$\Rightarrow$ We can always find a λ which satisfies $\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0$ condition (for $\vec{\nabla}\cdot\vec{A}_0\to 0$ at ∞. For the other case we need to change our strategy to find λ but we will not touch upon it here).

☆ It is always possible to make vector potential divergenceless.

(**) $\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$ Poisson Eq.

$\Rightarrow$ Actually, above eq. contains 3 Poisson Eq.s for each Cartesian coordinate.

$\nabla^2\vec{A} = (\nabla^2 A_x)\hat{x} + (\nabla^2 A_y)\hat{y} + (\nabla^2 A_z)\hat{z} \Rightarrow \boxed{\nabla^2 A_i = -\mu_0 J_i}$ $i:1,2,3$ or (x,y,z)

☆ If $\vec{J}\to 0$ at infinity, the solution for (**) is given by

$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{J(\vec{r}')}{\mathscr{r}}d\tau'}$ for volume currents

$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int\frac{I}{\mathscr{r}}dl' = \frac{\mu_0 I}{4\pi}\int\frac{1}{\mathscr{r}}d\vec{l}'}$ for line currents

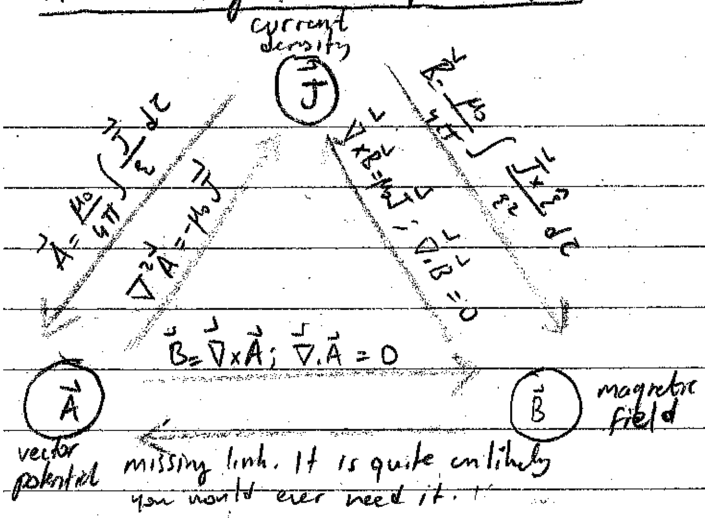
$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{K}}{\mathscr{r}}da'}$ for surface currents

☆ $\vec{A}$ isn't as useful as V: It is still a vector; we need to handle 3 components. It is easier to work with above equations than working with Biot-Savart law but still components make this process hard.

It would be nice if we could come up sth like $\vec{B}=-\vec{\nabla}U$ but this makes no sense since $\vec{\nabla}\times\vec{B} = -\vec{\nabla}\times(\vec{\nabla}U)=0$ is always true in vector calculus.

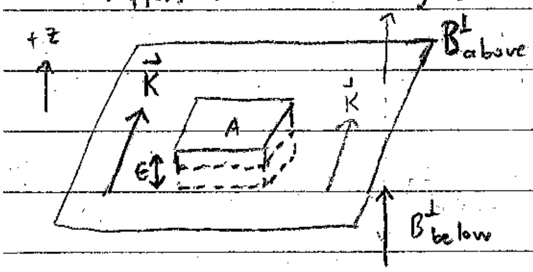
☆ Most of the time $\vec{A}$ has a substantial theoretical importance.

A summary for magnetostatics



Magnetostatic Boundary Conditions

Just as $\vec{E}$ suffers a discontinuity at a surface charge, $\vec{B}$ suffers a discontinuity at a surface current.

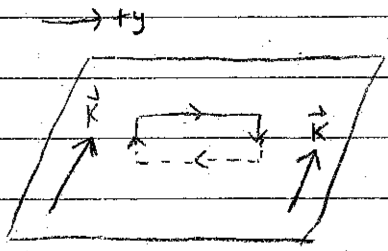


Since $\nabla\cdot\vec{B} = 0$ $\quad \oint \vec{B}\cdot d\vec{a} = 0$

$\Rightarrow B^{\perp}_{above} = B^{\perp}_{below}$

$\lim_{\epsilon\to 0} \oint \vec{B}\cdot d\vec{a} = B^{\perp}_{above} A - B^{\perp}_{below} A = 0$

$\perp$ component

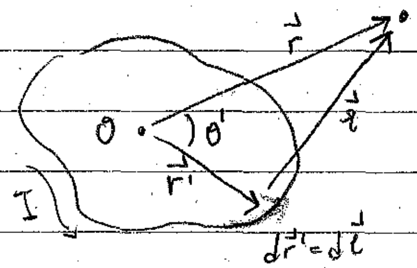


$\oint \vec{B}\cdot d\vec{l}$

$\parallel$ component

Multipole Expansion of the Vector Potential

- We would like to have an approximate formula for $\vec{A}$ of a localized current distribution, valid at distant point.
- The idea is to write the potential in the form of a power series $1/r$.



$$\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathfrak{r}}\, d\vec{l}\,'$$

We showed before that

$$\frac{1}{\mathfrak{r}} = \underbrace{\frac{1}{\sqrt{r^2+(r'^2)-2rr'\cos\theta'}}}_{\text{cosine theorem}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\theta')}_{\text{Legendre Polinomial}}$$

Therefore, $\vec{A}$ for current loop above:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')\, d\vec{l}\,'$$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\theta')\, d\vec{l}\,'}$$

More explicitly:

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint d\vec{l}\,'}_{\text{Monopole term}} + \underbrace{\frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}\,'}_{\text{Dipole term}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)d\vec{l}\,'}_{\text{Quadrupole term}} + \dots\right]$$

★ Note that pure mathematics asserts ~~the~~ the fact that $\oint d\vec{l}\,' = 0$; since total displacement around a closed loop is zero. This reflects that there are no magnetic monopoles in nature. Remember that Biot-Savart law directly told us that $\vec{\nabla}\cdot\vec{B} = 0$ and from vector calculus we know that if $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$. Therefore we derived a suitable $\vec{A}$ and considered its multipol exp. It also turned out from here that $\vec{A}(\vec{r})$ contains no monopole term.

★ What if we considered an open loop so that $\int d\vec{l}\,' \neq 0$? Remember that an open wire case actually doesn't reflect a real physical situation since an open wire cannot hold a steady current. It can only be interpreted as a part of a larger circuit and if we calculate the Biot-Savart Law for it, we could get the contribution of of it.

- In absence of any monopole contribution

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta'\, d\vec{l}\,' = \frac{\mu_0 I}{4\pi r^2}\underbrace{\oint (\hat{r}\cdot\vec{r}\,')\, d\vec{l}\,'}_{-\hat{r}\times\int d\vec{a}\,'}$$

HW 1.60(e) 1.61(e)

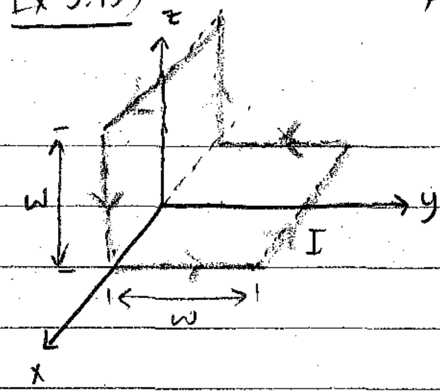
$$= \frac{\mu_0}{4\pi r^2}\left[-\hat{r}\times\underbrace{\left(I\int d\vec{a}\,'\right)}_{\equiv \vec{m}}\right]$$

magnetic dipole moment. and $\vec{a} = \int d\vec{a}\,'$ is vector area

direction is assigned acc. to Right hand rule.

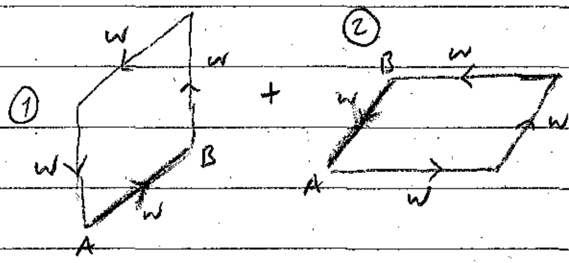
$$= \frac{\mu_0}{4\pi r^2}\left(I\int d\vec{a}\,'\right)\times\hat{r} = \frac{\mu_0}{4\pi r^2}\left(I\vec{a}\right)\times\hat{r} \Rightarrow \vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

Ex 5.13)



Find the $\vec{m}$.

This wire can be considered the superposition of two plane square loops.



* Extra sides AB cancel each other when 2 are put together.

- Magnitude for ① $|\vec{m}| = Iw^2$ • Magnitude for ② : $|\vec{m}| = Iw^2$
- Direction for ① $\hat{y}$ • Direction for ② . $\hat{z}$

$$\Rightarrow \vec{m} = Iw^2(\hat{y}+\hat{z})$$

* $\vec{m} = I\vec{a}$ is independent from the choice of origin contrary to electric dipole moment; $\vec{p}$ is independent of the origin only when total charge vanishes. Since magnetic monopole moment is always zero, $\vec{m}$ is always independent of origin.

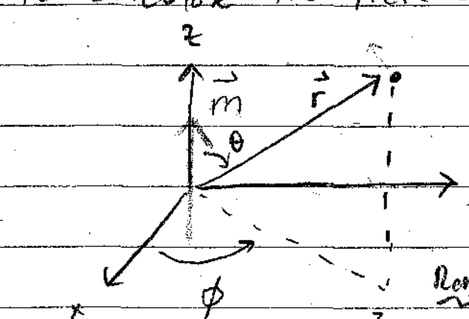
- Remember: $\underbrace{V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}_{\text{Dipole contribution}}$ $\quad \underbrace{\vec{p} \equiv \int \vec{r}'\rho(\vec{r}')d\tau'}_{\text{electric dipole moment}} \Rightarrow$ Depends on the choice of origin.

$\vec{m} = I\vec{a} \Rightarrow$ Doesn't depend on the choice of origin.

* Dipole cont. dominates but there are other cont. from other higher order terms. Dip. cont. is a good approx. to the true pot. but it isn't exact. Can we think about an "pure" dipole as we did for electric dipole? We can theoretically do it but physically it is not possible.

→ Think of an infinitesimally small loop at the origin but to keep the dipole moment finite, we need to increase the current to infinity, so that the product $m = Ia$ is held constant. In practice, the dipole potential is a suitable app. wherever the distance r greatly exceeds the size of the loop.

- To calculate the field of a pure magnetic dipole.

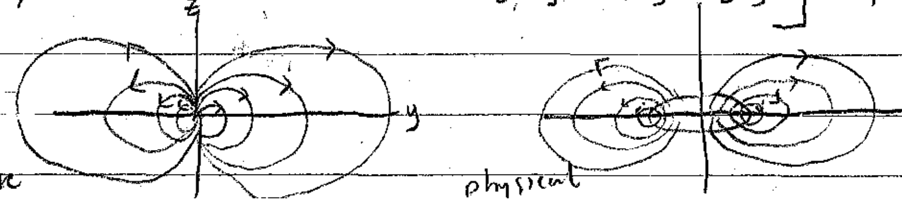


$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla}\times\vec{A} = \frac{\mu_0 m}{4\pi}\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}A_\phi\sin\theta\,\hat{r} - \frac{\partial}{\partial r}(rA_\phi)\hat{\theta}\right.$$

Remember: $\boxed{\vec{\nabla}\times\vec{A} = [\dots]\hat{r} + [\dots]\hat{\theta} + [\dots]\hat{\phi}}$

$$\vec{B}_{dip}(\vec{r}) = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

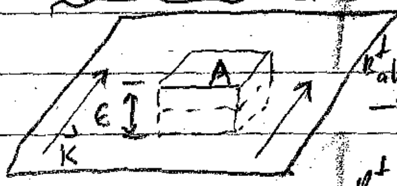


pure physical

Magnetostatic Boundary Conditions

- Magnetic field is discontinuous at a surface current just like electric field.

Perpendicular component. (for any surface)



If there is any perpendicular component of $\vec{B}$ to the surface it must have the same direction above $\rightarrow +y$ and below the surface. Why?

$\rightarrow$ Symmetry: If they would be in opposite direction it would make no sense; why?

- Reverse $\vec{K}$: Then, acc. to Biot-Savart law both $B^{\perp}_{above}$ and $B^{\perp}_{below}$ must change direction.

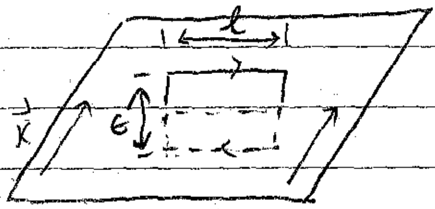
But, reversing $\vec{K}$ is equivalent to rotating the surface $180°$ about y axis which wouldn't change the fact that if $\vec{B}$ is into the surface it will still be into the surface (or if it is out of the surface it will be out). This contradicts above statement.

$$\vec{\nabla} \cdot \vec{B} = 0 \rightarrow \oint \vec{B}\, d\vec{a} = 0$$

$$B^{\perp}_{above} A - B^{\perp}_{below} A = 0$$

$$\boxed{B^{\perp}_{above} = B^{\perp}_{below}}$$ $\perp$ component is continuous!

Parallel component



$$\oint \vec{B} \cdot d\vec{\ell} =$$