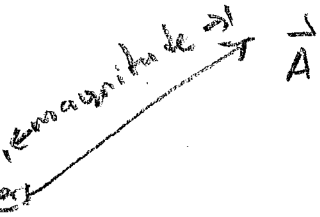
Vectors and vector operations

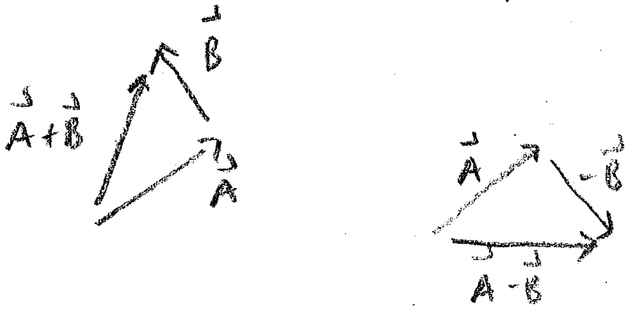
Independent quantities in space having a magnitude and direction.

They are represented by arrows:



operation on vectors

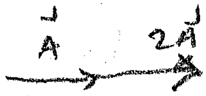
i) Can be added and subtracted.



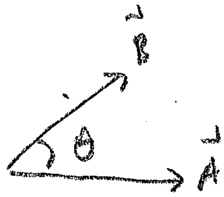
ii) Multiplication by a scalar.

$a(\vec{A}+\vec{B}) = a\vec{A} + a\vec{B}$; only rescales the vector does not change its direction.

The result is a vector.



iii) Dot product



$\vec{A}.\vec{B} \equiv AB\cos\theta$ → The result is a scalar!

$= \vec{B}\cdot\vec{A}$; commutative (by definition)

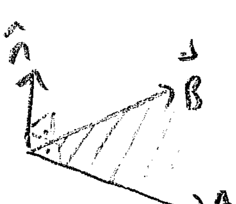
$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

- $\vec{A}.\vec{A} = A^2$
- $\vec{A}.\vec{B} = \begin{cases} AB, & \vec{A} \parallel \vec{B} \\ 0, & \vec{A} \perp \vec{B} \end{cases}$

iv) Cross product

$\vec{A}\times\vec{B} \equiv AB\sin\theta\, \hat{n}$

$\hat{n}$: unit vector (length 1)



Right-hand rule.

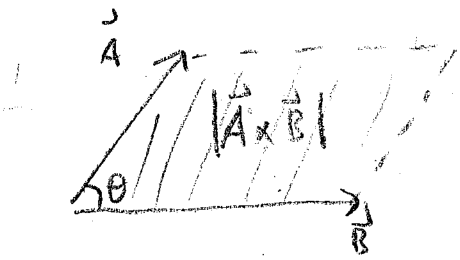
$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$

Distributive

but not commutative

$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

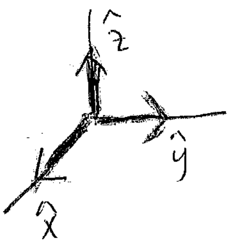
Geometrically:



- $\vec{A}\times\vec{B} = \begin{cases} 0 , & A \parallel B \\ AB , & A \perp B \end{cases}$

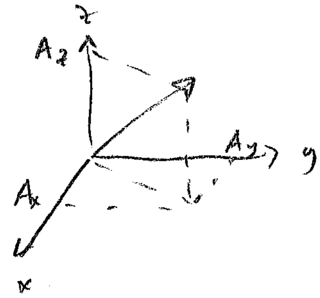
- $\vec{A}\times\vec{A} = 0$

Component form:



$\hat{x},\hat{y},\hat{z}$; unit vectors
$|\hat{x}| = |\hat{y}| = |\hat{z}| = 1$
$\hat{x}\perp\hat{y}$, $\hat{y}\perp\hat{z}$, $\hat{x}\perp\hat{z}$

$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$



i) Addition

$$\vec{A}+\vec{B} = (A_x+B_x)\hat{x} + (A_y+B_x)\hat{y} + (A_z+B_z)\hat{z}$$

ii) Mult. by a scaler

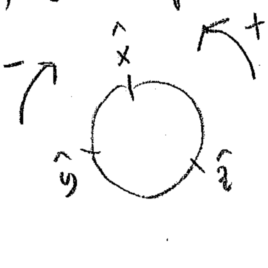
$$c\vec{A} = cA_x\hat{x} + cA_y\hat{y} + cA_z\hat{z}$$

iii) Dot product

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2 \rightarrow A = \sqrt{\vec{A}\cdot\vec{A}}$$

iv) Cross product.

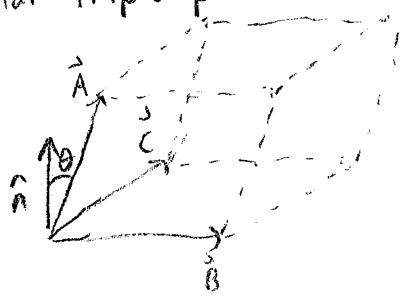


$$\vec{A}\times\vec{B} = (A_zB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Triple Products

i) Scalar triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$; Scalar (geometrically, volume of parallelepiped)



$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Alphabetical order is preserved

$$\vec{A}\cdot(\vec{C}\times\vec{B}) = \vec{B}\cdot(\vec{A}\times\vec{C}) = \vec{C}\cdot(\vec{B}\times\vec{A})$$

They have ~~negative~~ opposite sign.

iii) Vector triple product. $\vec{A}\times(\vec{B}\times\vec{C})$ BAC-CAB rule

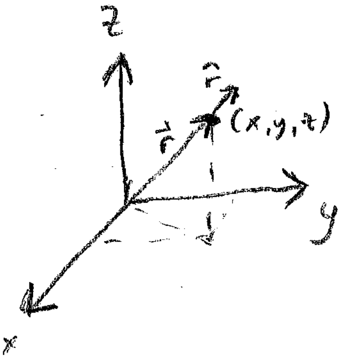
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - C(\vec{A}\cdot\vec{B})$$

$$(\vec{A}\times\vec{B})\times\vec{C} = -\vec{C}\times(\vec{A}\times\vec{B}) = -\vec{A}(\vec{C}\cdot\vec{B}) + \vec{B}(\vec{C}\cdot\vec{A})$$

A B C

completely different.

Position, displacement and Seperation Vectors



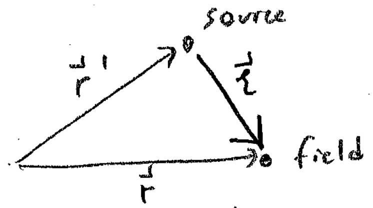
$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ position vector

$\hat{r} = \dfrac{\vec{r}}{r}$: unit vector pointing radially outwards.

$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$: infinitesimal displacement vector.

- In electrodynamics we usually deal with problems involving 2 points

Source point $\vec{r}'$ and field point $\vec{r}$



we define the seperation vector as

$\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$ $\mathscr{r}$: magnitude $\hat{\mathscr{r}} = \dfrac{\vec{\mathscr{r}}}{\mathscr{r}}$ unit vector from source point to field point

Before this explain sceuler and vector fields $\mathscr{r}$ in detail.

Maybe I better change this with $\vec{R}$.

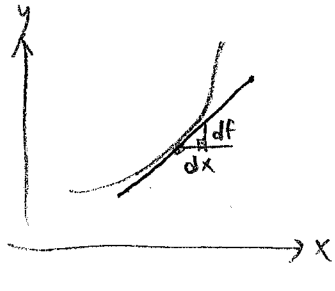
Differential Calculus.

$f(x)$ How function varies as we change its argument x by a tiny amount

differensial or $df = \left(\dfrac{df}{dx}\right)dx$
of $f(x)$

Derivative of $f(x)$ w.r.t. x.

Geometrical interpretation;



$\dfrac{df}{dx}$ gives a function providing the slope of the curve.

Gradient: Assume that we have a function of 3 variables instead of 1.

$T(x,y,z)$: e.g. it gives the temperature of a room in every point x,y,z.

we want to generalize the notion of derivative for the functions with 3 variables.

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$= \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)\cdot\left(dx\hat{x} + dy\hat{y} + dz\hat{z}\right)$$

$$= \vec{\nabla}T \cdot d\vec{l}$$

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$ gradient of T. (vector quantity)

Geometrical Interpretation:

$$dT = \vec{\nabla}T \cdot d\vec{l} = |\vec{\nabla}T|.|d\vec{l}|\cos\theta = |\vec{\nabla}T|\cos\theta$$

dT is greatest when $\theta = 0$, i.e, when we move in the vector field in the same direction with $\vec{\nabla}T$ (the gradient). Therefore,

- $\vec{\nabla}T$ points in the direction of maximum increase of the func. T.
- $|\vec{\nabla}T|$ gives the slope in max. inc. direction.

e.g: $f(x,y) = x^2 + y^2$

★ Pouring water over a hill example $H(x,y)$

The Operator $\vec{\nabla}$:

$\vec{\nabla}$ is actually an operator and can act in three ways on fields:

$$\vec{\nabla}T = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)T$$

1. On a scaler function
2. On a vector function via dot product
3. On a " " via cross product

★ We will see that usage of $\vec{\nabla}$ is very useful in formulating electromagnetic theory. It saves a lot of space.

The Divergence (Dot product of $\vec{\nabla}$ with a vector function)

$$\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(V_x\hat{x} + V_y\hat{y} + V_z\hat{z}\right)$$

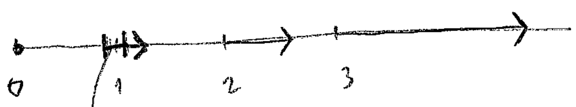
$$\underbrace{\vec{V}(x,y,z)}_{\text{vector function}} \xrightarrow[\text{to}]{\text{Transforms}} \underbrace{\vec{\nabla}\cdot\vec{V}(x,y,z)}_{\text{Scalar function}}$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$: It is a scalar.

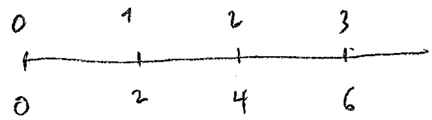
→ How much the vector $\vec{V}$ spreads out (diverges) from the point in question

For example:

$\vec{f}(x) = x^2\hat{x} \rightarrow \vec{\nabla}\cdot\vec{f} = 2x$



$\vec{f}(x)$

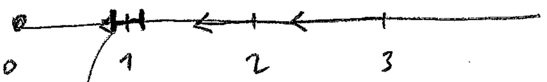


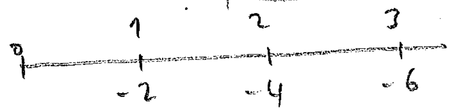
$\vec{\nabla}\cdot\vec{f}(x)$

If we think about a little line segment. The flux getting in is smaller than the flux getting out. Therefore, if we think about that small region covered by that little line segment there is a net flux going out from there. This is positive divergence. And this is true for all the other points in this linear space (shift the line segment to the right and the divergence will be still positive and increasing)

Flux: Rate of flow of a property per unit area.

If we had $\vec{f}(x) = -x^2\hat{x}$ and $\vec{\nabla}\cdot\vec{f} = -2x$

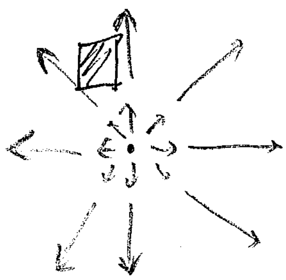




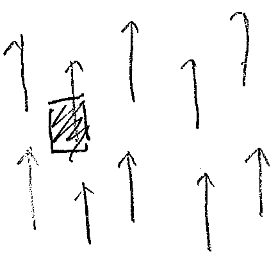
Now the flux getting into this line segment is smaller than that of getting out. So there is a net flux getting into this little segment. This is a negative divergence. If it was a matter flux than that matter would accumulate in that region

other examples in 2D:

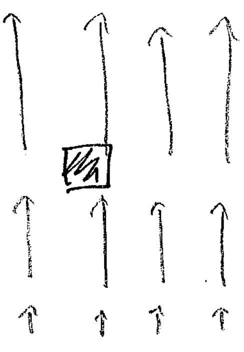
Think about a small area element.



Positive div.



Zero div.



Positive div.

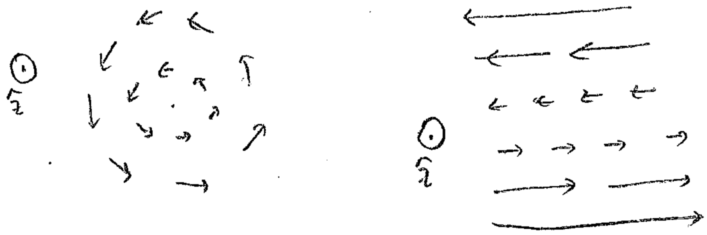
The Curl (Cross product of $\vec{\nabla}$ with a vector function)

From the definition of cross product and $\vec{\nabla}$ we can construct curl as:

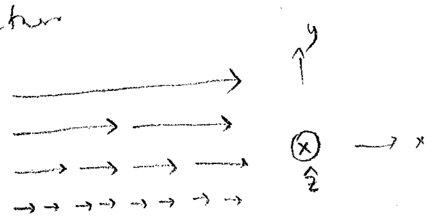
$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} = \hat{x}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{y}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{z}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

It is a measure of how much $\vec{V}$ curls around the point in question.

Ex 1) Think about the surface in a pool. When you drop a tooth pick on the surface, if it starts to rotate then the point you placed it is has a nonzero curl.



These 2D functions have a curl in $\hat{z}$ direction



In Summary:

$\vec{\nabla} f$: yields a vector function.
$\vec{\nabla}\cdot\vec{V}$: yields a scalar function.
$\vec{\nabla}\times\vec{V}$: yields a vector function.

Second Derivatives: There is one more operation that we will be using during this class. It is Laplacian

It can be thought in terms of

$$\vec{\nabla}\cdot\vec{\nabla}T = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot\left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$$

$$= \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)T = \nabla^2 T$$

Some identities about second derivatives

- $\vec{\nabla}\times(\vec{\nabla}T) = 0$ The Curl of gradient is always zero.
- $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$ The divergence of curl is always zero

HW Prove them briefly

More comments on Divergence, curl and $\vec{\nabla}\cdot(\vec{\nabla}\times V)=0$

Let us modify $\vec{V}_a = -y\hat{x}+x\hat{y}$ to make things clearer for $\vec{\nabla}\cdot$ and $\vec{\nabla}\times$

Consider the curls of following 3 cases (and then their divergences).

i) $\vec{V}_a = \frac{1}{\sqrt{x^2+y^2}}(-y\hat{x}+x\hat{y})$

ii) $\vec{V}_b = -y\hat{x}+x\hat{y}$

iii) $\vec{V}_c = -yz\hat{x}+xz\hat{y}$

? iv) $\vec{V}_d = (-y+x)\hat{x}+(x+y)\hat{y}$ ⇝ I will think about this.

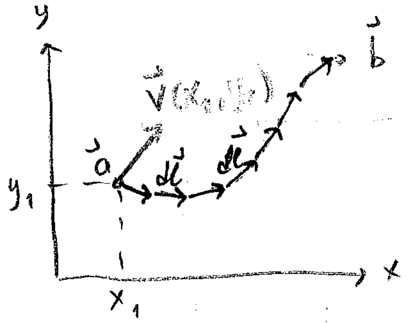
Integral Calculus (We will be using line, surface and volume integrals quite often in this class)

Line Integral

$\int_{\vec{a}\,P}^{\vec{b}} \vec{v} \cdot d\vec{l}$ It is calculated from point $\vec{a}$ to $\vec{b}$ along a path P

$\oint \vec{v} \cdot d\vec{l}$ If the path forms a closed loop ($\vec{b} = \vec{a}$) we put a circle on the integral sign.

How to evaluate?



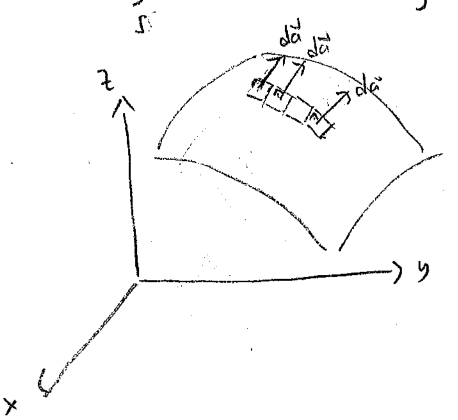
At each point on the path we take the dot product of $\vec{v}$ (evaluated at that point) with the displacement $d\vec{l}$. And we add all these results along the path.

REMEMBER: Most famous line integral in mech. class was $W = \int \vec{F} \cdot d\vec{l}$ if ~~this~~ the result of this integral was independent from the path $\vec{F}$ was conservative.

Surface Integral

$\int_S \vec{v} \cdot d\vec{a}$ or $\oint \vec{v} \cdot d\vec{a}$

It is calculated over a surface S. We put a circle on the integral if the surface is closed.



$d\vec{a}$ infinitesimal area with a direction $\perp$ to the surface at that point

How to calculate:

Calculate every $\vec{v} \cdot d\vec{a}$ and add them up.

* If $\vec{v}$ describes the flow of a fluid, (mass per unit area per unit time) then $\int \vec{v} \cdot d\vec{a}$ represents the total mass per unit time passing through the surface, we call it <u>flux</u>.

Volume Integrals $\int_V T\,d\tau$: T scalar function

$d\tau$ infinitesimal volume element.

$d\tau = dx\,dy\,dz$ (in cartesian coordinates).

For example: T : density of a substance

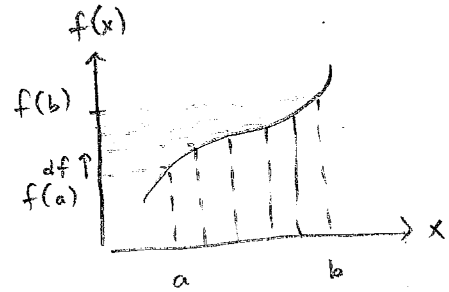
Then $\int_V T\,d\tau$ would give the total mass.

The Fundamental Theorem of Calculus.

$$\int_a^b F(x)\,dx = f(b) - f(a) \qquad F(x) = \frac{df}{dx}$$

⇝ It links the concept of differentiating a function with the concept of integrating a function.

Geometrical Interpretation:



$$\int_a^b \frac{df}{dx}\,dx$$

df : infinitesimal change in f when we move from x to $x+dx$

we are adding all these changes up.

It is obvious that it is exactly the same to add up all df's and to substract $f(a)$ from $f(b)$.

★ The integral of a derivative over an interval is given by the value of the function at the end points (boundaries).

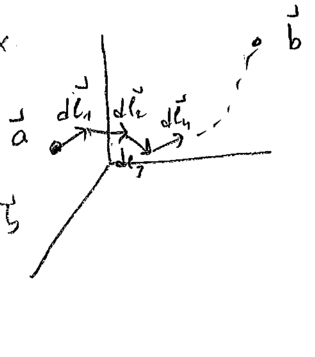
★ Similarly in vector calculus there are three species of derivative (gradient, divergence, curl) and each has its own fundamental theorem.

The Fundamental Theorem for Gradients.

Assume we have a function of 3 variables $T(x,y,z)$. If we start from a point $\vec{a}$ and move a small distance $d\vec{l}_1$, the function will change

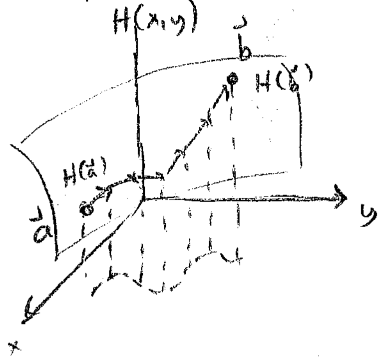
$$dT = (\vec{\nabla}T)\cdot d\vec{l}_1 \xleftarrow{\text{analogy}} df = f'dx$$

If we take another step $d\vec{l}_2$, T will increase by $(\vec{\nabla}T)\cdot d\vec{l}_2$. In this manner we can reach up to a point $\vec{b}$. Therefore, the total change from $\vec{a}$ to $\vec{b}$ along a path P



$$\boxed{\int_{\vec{a}\,P}^{\vec{b}} (\vec{\nabla}T)\cdot d\vec{l} = T(\vec{b}) - T(\vec{a})}$$ → Fundamental Theorem for gradients

If it is hard to think about it for a function with 3 variables. think of a height function $H(x,y)$



You can climb this hill step by step or you can put some altimeters at points $\vec{a}$ and $\vec{b}$ and substract the altimeter readings.

<u>Fundamental Theorem for **Divergences**.</u> (Gauss theorem, Green's theorem divergence theorem)

$$\int_{\mathcal{V}} (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$$

$\vec{\nabla}\cdot\vec{v}$ — volume $\mathcal{V}$

★ Again it simply states that the <u>integral of a derivative</u> over a region is equal to the value of the function at the boundary

$\underbrace{\text{boundary}}_{\text{the surface that bounds the volume}}$

→ Now boundary term itself is an integral. : Reasonable ~ boundary of a line is just two end points. But the boundary of a volume is a closed surface.

<u>Geometrical Interpretation :</u>

$\vec{v}$ represents the flow of an incompressible fluid.

$\oint \vec{v}\cdot d\vec{a}$: total amount of fluid passing through the surface per unit time.

- V can be thought in the units of $\frac{\text{sth}}{\text{area time}}$ ~> flux.

$\vec{\nabla}\cdot\vec{v}$: $\frac{\text{sth}}{\text{volume time}}$

$\underbrace{\int(\text{faucets within the volume})}_{\text{summing the differential flux over the volume.}} = \underbrace{\oint(\text{flow out through the surface})}_{\text{summing the differential flux over the surface wrapping the region}}$

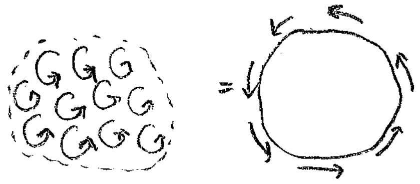
The Fundamental Theorem for Curls (Stokes' theorem)

$$\int_S (\vec{\nabla}\times\vec{V})\cdot d\vec{a} = \oint_P \vec{V}\cdot d\vec{l}$$

The integral of a derivative (curl) over a region (a patch of surface) is equal to the value of the function at the boundary (perimeter of the patch)

Geometrical interpretation:

$\vec{V}$: in units of $\frac{stb}{length\ time}$



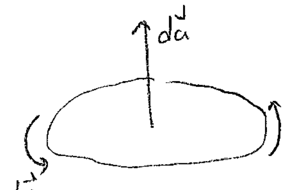
$\vec{\nabla}\times\vec{V}$: in units of $\frac{stb}{area\ time}$

$\int_S (\vec{\nabla}\times V)\cdot d\vec{a}$: Sum up differential swirls over an area

$\oint_P \vec{V}\cdot d\vec{l}$: How much flow is following the boundary.

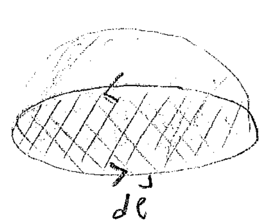
Consistency about the sign in Stokes' theorem:

For a closed surface $d\vec{a}$ point in the direction of outward
what about an open surface ?



The consistency is obtained with the RHR.

For which surface will the integral $\int_S (\vec{\nabla}\times V)\cdot d\vec{a}$ be evaluated?



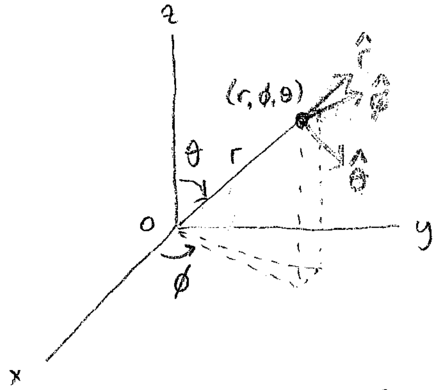
For the same line integral, we can define infinite number of surfaces on where the surface integral be taken.

Since the theorem does not refer to any particular surface. we can come up with 2 corrolaries for this theorem:

- Corrolary 1: $\int (\vec{\nabla}\times\vec{V})\cdot d\vec{a}$ depends only on the bounding line, not on the particular surface used.
- Corrolary 2: $\oint (\vec{\nabla}\times\vec{V})\cdot d\vec{a} = 0$ for any closed surface because the boundary line shrinks down to a point when we closed the surface, and hence $\oint_P \vec{V}\cdot d\vec{l}$ vanishes.

Curvilinear Coordinates → Start with polar coordinates.

1) Spherical coordinates



r : distance from the origin $[0, \infty)$

ϕ : polar angle $[0, 2\pi]$

θ : azimutal angle $[0, \pi]$

$\vec{r}, \vec{\phi}, \vec{\theta}$: unit vectors

$\vec{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_\theta \hat{\theta}$

In cartesian coordinates

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi$: infinitesimal displacement.

$d\tau = r^2 dr\, d\theta \sin\theta$: volume element.

- Area element depends on the surface we are considering. If we are integrating over a sphere ($r =$ constant)

$\Rightarrow d\vec{a} = r^2 \sin\theta\, d\phi\, d\theta\, \hat{r}$

- if the surface lies in the xy plane. ($\theta = \pi/2$)

$\Rightarrow d\vec{a} = r\,dr\,d\phi\,\hat{\theta}$

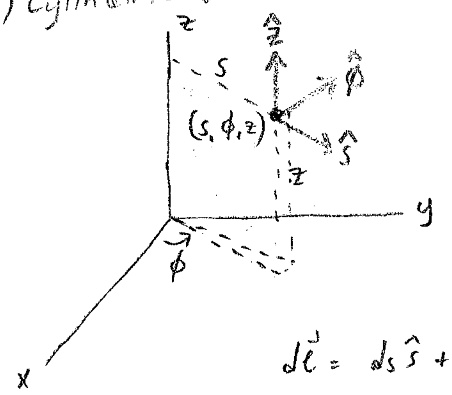
- $\vec{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$: Gradient

- $\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$: Divergence.

HW Show them

- $\vec{\nabla} \times \vec{v} = \ldots$: curl

2) Cylindrical Coordinates.



s : distance from z-axis

ϕ : angle from x-axis

z : height.

$\hat{s}, \hat{\phi}, \hat{z}$: unit vector.

$\vec{A} = A_s \hat{s} + A_\phi \hat{\phi}, A_z \hat{z}$

In cartesian coordinates:

$x = s\cos\phi$

$y = s\sin\phi$

$z = z$

$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$: infinitesimal displacement.

$d\tau = s\,ds\,d\phi\,dz$: volume element.

$\vec{\nabla} T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$ Gradient

$\vec{\nabla} \cdot \vec{T} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$ Divergence

HW show them.

$\vec{\nabla} \times \vec{T} = \ldots$ Curl

The Dirac Delta Function

- The issue starts with an innocent-looking vector function:

$$\boxed{\vec{V} = \frac{1}{r^2}\hat{r}} \longrightarrow$$ Let us apply divergence theorem for this function.

$$\int_V (\vec{\nabla}\cdot\vec{V})\, d\tau = \oint \vec{V}\cdot d\vec{a}$$

Evaluate it over a sphere with radius R.

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = 0 \quad \text{and} \quad \underbrace{\oint\left(\frac{1}{\cancel{R^2}}\hat{r}\right)\cdot\left(\cancel{R}\, d\theta\, R\sin\theta\, d\phi\, \hat{r}\right)}$$

$$= \int_0^{\pi}\int_0^{2\pi}\sin\theta\, d\theta\, d\phi = \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi = -\left[\cos\theta\right]_0^{\pi} 2\pi = 4\pi$$

$\Rightarrow 4\pi \neq 0$! : Is divergence theorem is false?

First of all notice that divergence is not defined for $r \leq 0$. $1/r^2$ has an asymptote at $r=0$, it explodes. Therefore, whatever we obtain after $\vec{\nabla}\cdot\vec{V}$ is valid for $r>0$. It may look weird but $\vec{\nabla}\cdot\vec{V} = 0$ says that the divergence of this vector field is zero for $r>0$. So, there is no net ~~flux~~ divergence in any region of the space "covered" with this vector field. Be also aware of that this is the only power function having this feature:

$$\vec{\nabla}\cdot\left(\frac{1}{r}\hat{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^{\cancel{2}}\frac{1}{\cancel{r}}\right) = \frac{1}{r^2} \quad : \text{net positive divergence}$$

$$\vec{\nabla}\cdot\left(\frac{1}{r^3}\hat{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^3}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = (-1)\frac{1}{r^4} \quad : \text{net negative divergence.}$$

It is obvious that $\vec{\nabla}$ cannot give any information about the point $r=0$, where $1/r^2$ explodes. On the other hand, $\oint \vec{V}\cdot d\vec{a} = 4\pi$ tells us that, if the divergence theorem is true, we should get a flux of 4π for any sphere with radius R, it does not matter how small the radius is. Therefore, the whole contribution 4π must come from the origin, $r=0$. Thus, $\vec{\nabla}\cdot\vec{V}$ has a bizarre property that it vanishes everywhere except the point $r=0$. And yet its integral (over any volume containing that point) is 4π. No ordinary function behaves in that way.

A physical example could refer to a density of a point particle: It is zero except at the exact location of the particle, yet its integral at that point is still finite. (The density has an asymptote where the particle is located but it still has a finite mass at that particular point.)

$$\int \rho\, d\tau = \text{finite} \qquad \rho = \begin{cases}\infty, & r = 0\\ 0, & r \neq 0\end{cases} \qquad \rho: \text{density.}$$

To get rid of this inconsistency we make use of a special function called Dirac Delta Function.

One dimensional Dirac Delta Function $\delta(x)$

$$\delta(x) \equiv \begin{cases} 0 , & \text{if } x \neq 0 \\ \infty , & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x)\, dx \equiv 1$$

- Technically $\delta(x)$ is not a mathematical function because it has no finite value at $x=0$. It is actually known as generalized function, or distribution.

- One important feature of the $\delta(x)$ is:

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)$$: without $f(x)$ the integral would yield 1. Since we have the multiplication $f(x)\delta(x)$, the integral "picks up" that particular value of $f(x)$ at $x=0$.

- We can generalize $\delta(x)$ by shifting the spike from $x=0$:

$$\delta(x-a) = \begin{cases} 0 , & x \neq a \\ \infty , & x = a \end{cases} \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x-a) f(x)\,dx = f(a)$$

Three dimensional Dirac Delta Function $\delta^3(\vec{r})$

$$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z) \qquad \int_{\text{all space}} \delta^3(\vec{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

$$\int_{\text{all space}} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,d\tau = f(\vec{a})$$

- So how to resolve the $4\pi \neq 0$ paradox we encountered for the vector funct. $\frac{1}{r^2}\hat{r}$?

  i) $\vec{\nabla}\cdot\vec{V} = 0$ everywhere except origin

  ii) Yet its integral over any volume containing the origin is a constant (4π)

  → These precisely define a Dirac Delta function:

$$\vec{\nabla}\cdot\left(\frac{1}{r^2}\,\hat{r}\right) = 4\pi\,\delta^3(\vec{r})$$

The Theory of Vector Fields

The Helmholtz Theorem:

After Faraday, the laws of electricity and magnetism expressed by $\vec{E}$ (electric field) $\vec{B}$ (magnetic field), which are nothing but some vector functions. We will see that these are some differential equations of $\vec{E}$ and $\vec{B}$ involving (not surprisingly) divergence and curl. Maxwell later boiled down the whole theory into 4 equations with divergence and curl of $\vec{E}$ and $\vec{B}$.

Question: To what extent is a vector function determined by its divergence and curl?

$$\vec{\nabla}\cdot\vec{F} = D \qquad \vec{\nabla}\times\vec{F} = \vec{C} \quad \Rightarrow \quad \text{can we determine } \vec{F}\text{?}$$

Not generally. We can do this only if we specify some appropriate boundary conditions. In electrodynamics, we require that the fields go to zero "at infinity" (far away from all charges) under this condition a vector field can be uniquely decomposed into the sum of an irrotational and a solenoidal vector field.

(sufficiently smooth, rapidly decaying vector field)

$$\vec{F} = \underbrace{-\vec{\nabla}V + \vec{\nabla}\times\vec{A}}_{\text{irrotational}} \longrightarrow \text{solenoidal}$$

Potentials:

We have already seen that $\int_{\vec{a}\,P}^{\vec{b}} (\vec{\nabla}T)\cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$: the result is independent from the path; it only depends on the boundary values.

Therefore if we have a vector function $\vec{F}$, which can be written as the gradient of a scalar function as $\vec{F} = -\vec{\nabla}V$ (minus is just conventional) then

$\int_{a}^{\vec{b}} \vec{F}\cdot d\vec{l}$ is independent of path for any given end points.

- We also know (you proved it in your HW) that $\vec{\nabla}\times\vec{F} = -\vec{\nabla}\times\underbrace{(\vec{\nabla}V)}_{0} = 0$
- Also, $\oint_P (\vec{\nabla}T)\cdot d\vec{l} = 0$ since starting and ending points are the same.

$\Rightarrow$ Theorem 1) If we have a vector field of no curl $\vec{\nabla}\times\vec{F} = 0$ (curl-less or irrotational field) (or maybe more easily, if we have a vector field $(\vec{F})$ which can be written as the gradient of a scalar funct.) the following conditions are equivalent:

a) $\vec{\nabla}\times\vec{F} = 0$

b) $\int_a^b \vec{F}\cdot d\vec{l}$ = independent of path for a given pair of $\vec{a}$ and $\vec{b}$

c) $\oint \vec{F}\cdot d\vec{l} = 0$ for any close loop

d) $\vec{F} = -\vec{\nabla}V$

★ • Note that the scalar function V is not unique; any constant can be added to V.

- $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ (proved in the HW). Therefore following statements are equivalent:

Theorem 2) Divergence-less (or solenoidal) fields. $(\vec{F})$

a) $\vec{F} = \vec{\nabla}\times\vec{A}$

b) $\vec{\nabla}\cdot\vec{F} = 0$ everywhere.

c) $\int_S \vec{F}\cdot d\vec{a}$ is independent of the surface for a given boundary line (Stokes' theorem)

d) $\oint \vec{F}\cdot d\vec{a} = 0$ for any closed surface.

- Note that $\vec{A}$ is not unique. The gradient of any scalar function can be added to it since the curl of a gradient is zero.

★ Note that in all cases above the vector field $\vec{F}$ can be written as a summation of a gradient of a scalar function plus the curl of a vector function

$$\boxed{\vec{F} = -\vec{\nabla}V + \vec{\nabla}\times\vec{A}}$$

- If we turn back to Helmholtz theorem V and $\vec{A}$ are actually D and $\vec{C}$ respectively. Helmholtz theorem states that we can uniquely define an $\vec{F}$ once we are given a scalar function V and a vector function $\vec{A}$ with appropriate boundary conditions..