Proper time and proper velocity

Assume that we are moving in a train, and the clock in the station ticks off an interval dt, our watch only advances.

$$d\tau = \sqrt{1-\frac{v^2}{c^2}}\, dt \qquad v\text{: train's velocity w.r.t the station.}$$

$d\tau \rightarrow$ proper time.

$\boxed{\text{Proper time:}}$ The time registred by <u>our</u> watch.

- Assume that we are in a plane and the pilot announces that the plane's velocity is, for example, $\frac{c}{2}$, due South. What he means is that the velocity of the plane relative to ground

$$\vec{u} = \frac{d\vec{l}}{dt} \quad \begin{matrix} \rightarrow \text{distance covered on relative to ground} \\ \rightarrow \text{time registred by ground clocks.} \end{matrix}$$

It is the number we need to consider if we are concerning about being on time for an appointment in the destination (ordinary velocity.

- However, all we consider is the time we spend in the plane. proper velocity is more relevant

$$\vec{\eta} \equiv \frac{d\vec{l}}{d\tau} \quad \begin{matrix} \rightarrow \text{distance measured on the ground} \\ \rightarrow \text{time ~~measured~~ measured in the airplane.} \end{matrix}$$

$$d\tau = \sqrt{1-\frac{v^2}{c^2}}\, dt \rightarrow \frac{1}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{1}{dt} \rightarrow \frac{d\vec{l}}{d\tau} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{d\vec{l}}{dt}$$

$\frac{d\vec{l}}{d\tau}$: proper velocity; hybrid quantity. $\frac{d\vec{l}}{dt}$: ordinary velocity

Relativistic energy and Momentum

- Classical mechanics momentum = mass x velocity.

(?) which velocity are we going to chose : ordinary or proper.

$$\vec{\eta} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\vec{u} \rightarrow m\vec{\eta} = \frac{m\vec{u}}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \vec{p} = m\vec{\eta}$$

$m\vec{\eta}$: relativistic momentum.

Relativistic Dynamics

$\vec{F} = \frac{d\vec{p}}{dt}$ we choose to keep the force defined like this and take t as the "lab" time for 2 practical (and deeply physical) reasons:

i) Continuity with Newton's second law

classical correspondence $v \ll c$: $\vec{F}$ must smoothly reduce to Newtonian mechanics.

$$\vec{p} = m\vec{\eta} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \quad \frac{v^2}{c^2} \to 0 \qquad \frac{d\vec{p}}{dt} = m\frac{d\vec{\eta}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

ii) Operational definition of force

Forces are measured by how they change momentum per unit time as seen by the observer applying/measuring them. That observer's clock is the lab clock.

$$dW = \vec{F}\cdot d\vec{x} = \vec{F}\cdot\vec{u}\,dt = \frac{d\vec{p}}{\cancel{dt}}\cdot\vec{u}\,\cancel{dt} = \vec{u}\cdot d\vec{p}$$

$$dE_{kin} = \vec{u}\cdot d\vec{p}$$

Let us assume a 1D problem: $\qquad \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$

$$dE_{kin} = u\cdot d(\gamma m u)$$

$$d(\gamma u) = u\,d\gamma + \gamma\,du$$

$$\frac{d\gamma}{du} = \frac{d}{du}\left(1-\frac{u^2}{c^2}\right)^{-1/2} = \cancel{+}\frac{1}{\cancel{2}}\frac{1}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}\left(\frac{\cancel{2}u}{c^2}\right) = \frac{u}{c^2}\cdot\gamma^3$$

$$d(\gamma u) = \gamma\,du + u\frac{u\gamma^3}{c^2}du = \gamma\,du + \frac{u^2\gamma^3}{c^2}du$$

$$= \left(\gamma + \frac{u^2\gamma^3}{c^2}\right)du \qquad \gamma^2 = \frac{1}{1-\frac{u^2}{c^2}} \to \gamma^2 - \gamma^2\frac{u^2}{c^2} = 1$$

$$= \gamma\left(1 + \frac{u^2\gamma^2}{c^2}\right)du = \gamma^3 du \qquad \gamma^2 = 1 + \gamma^2\frac{u^2}{c^2}$$

$$d(\gamma m u) = m\,d(\gamma u) = m\gamma^3 du$$

$$dE_{kin} = u m \gamma^3 du$$

$$E_{kin}(u) = m\int_0^u \gamma^3 u' du' = m\int_0^u \frac{u'du'}{\left(1-\frac{u'^2}{c^2}\right)^{3/2}}$$

$\nu = 1 - \frac{u'^2}{c^2}$

$\frac{d\nu}{du'} = -\frac{2u'}{c^2}$

$c^2 d\nu = -2u'du'$

$u'du' = -\frac{1}{2}c^2 d\nu$

$$= m\int_{1}^{1-\frac{u^2}{c^2}} \left(-\frac{1}{2}\right)\frac{c^2 d\nu}{\nu^{3/2}}$$

$$= -\frac{c^2}{2}m\int \nu^{-3/2}d\nu = \cancel{-}\frac{c^2}{\cancel{2}}m \left.\frac{\nu^{-1/2}}{\cancel{\left(-\frac{1}{2}\right)}}\right|_1$$

$$= mc^2\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right)$$

$$E_{kin}(u) = mc^2(\gamma - 1)$$

Summary: Reconsider time-dilation expression:

moving reference frame. $\leftarrow$ $\Delta\bar{t} = \frac{1}{\gamma}\Delta t$ $\rightarrow$ station (or LAB) reference frame.

I will change the notation a little bit to define familiar quantities

$$d\tau = \sqrt{1-\frac{u^2}{c^2}}\,dt$$

$$\frac{1}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\frac{1}{dt} \xrightarrow[d\vec{l}]{\text{multiply by}} \underbrace{\frac{d\vec{l}}{d\tau}}_{\text{proper velocity}} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\underbrace{\frac{d\vec{l}}{dt}}_{\text{ordinary velocity}}$$

distance measured in LAB frame.

$$\boxed{\vec{\eta} = \gamma\vec{v}}$$

$$\xrightarrow[m]{\text{multiply by}} \quad m\vec{\eta} = \gamma m\vec{v} \equiv \vec{p} \quad \text{: relativistic momentum}$$

$$\xrightarrow{d/dt} \quad \frac{d}{dt}(m\vec{\eta}) = \frac{d}{dt}(\gamma m\vec{v}) = \frac{d\vec{p}}{dt} = \vec{F}$$

in the limit $u \ll c$

$$m\frac{d\vec{\eta}}{dt} = m\frac{d\vec{u}}{dt} = \frac{d\vec{p}}{dt}$$

- To define $\vec{F}$ we measure time in LAB frame. We have 2 practical and deeply physical reasons to do that:
  i) Continuity with Newton's 2nd law (as shown in the green box in the previous page)
  ii) Operational definition of force
  → Forces are measured by how they change momentum per unit time as seen by the observer applying/measuring them. ↓ That observer's clock is the LAB clock.

Relativistic work-energy theorem.

$$dW = \vec{F}\cdot d\vec{x} = \frac{d\vec{p}}{dt}\cdot \vec{u}\,dt = \vec{u}\cdot d\vec{p} \qquad \gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

For simplicity, let us examine 1D:

$$dE_{kin} = dW = u\,d(\gamma m u) = m u\, d(\gamma u)$$

$$d(\gamma u) = u\,d\gamma + \gamma\,du$$

Consider: $\frac{d\gamma}{du} = \frac{d}{du}\left(1-\frac{u^2}{c^2}\right)^{-1/2} = -\frac{2u}{c^2}\frac{1}{\left(1-\frac{u^2}{c^2}\right)^{3/2}}\left(-\frac{1}{2}\right) = \frac{u}{c^2}\gamma^3$

$$d\gamma = \frac{u}{c^2}\gamma^3 du$$

$$\Rightarrow d(\gamma u) = \gamma du + \frac{u^2}{c^2}\gamma^3 du = \left(\gamma + \frac{u^2}{c^2}\gamma^3\right)du = \gamma\left(\underbrace{1+\frac{u^2}{c^2}\gamma^2}\right)du$$

Note that: $\gamma^2 = \frac{1}{1-\frac{u^2}{c^2}} \to \gamma^2 - \gamma^2\frac{u^2}{c^2} = 1 \to \gamma^2 = \boxed{1+\gamma^2\frac{u^2}{c^2}}$

$$\Rightarrow d(\gamma u) = \gamma^3 du$$

$$dE_{kin} = m u \gamma^3 du \Rightarrow E_{kin}(u) = m\int_0^u \gamma^3 u' du' = m\int_0^u \frac{u'du'}{\underbrace{\left(1-\frac{u'^2}{c^2}\right)}_{\nu}{}^{3/2}}$$

$$\boxed{\begin{aligned}\frac{d\nu}{du} &= -\frac{2u'}{c^2}\\ c^2 d\nu &= -2u'du'\\ u'du' &= -\frac{1}{2}c^2 d\nu\end{aligned}}$$

$$\Rightarrow E_{kin}(u) = m\int_1^{1-\frac{u^2}{c^2}} \left(-\frac{1}{2}\right)c^2 d\nu\, \nu^{-3/2}$$

$$= -\frac{1}{2}mc^2\int_1^{1-\frac{u^2}{c^2}} \nu^{-3/2}d\nu = -\frac{\cancel{1}}{\cancel{2}}mc^2\frac{\nu^{-1/2}}{\cancel{\left(-\frac{1}{2}\right)}}\Bigg|_1^{1-\frac{u^2}{c^2}}$$

$$= mc^2(\gamma - 1) = \gamma mc^2 - mc^2$$

If the object is stationary ($E_{kin}=0$): $\qquad E_{tot} = E_{kin} + mc^2$

$\gamma mc^2 = mc^2 \equiv E_{rest}$: rest energy.

$$F_y = \frac{dp_y}{dt} \qquad \bar{F}_y = \frac{d\bar{p}_y}{d\bar{t}} = \frac{dp_y/dt}{\frac{1}{dt}\left(\gamma dt - \gamma \frac{v}{c^2} dx\right)} = \frac{F_y}{\gamma\left(1 - \gamma\frac{v}{c^2}\underbrace{\frac{dx}{dt}}_{u_x}\right)} = \frac{F_y}{\gamma\left(1-\gamma\frac{v}{c^2}u_x\right)}$$

$$\boxed{\begin{array}{l} \underline{\text{Lorentz transformations revisited}} \\ \bar{x} = \gamma(x - vt) \quad \rightarrow \quad d\bar{x} = \gamma dx - \gamma v dt \\ \bar{y} = y \\ \bar{z} = z \\ \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \quad \rightarrow \quad d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dx\right) \end{array}}$$

$$\frac{d\bar{x}}{d\bar{t}} = \gamma\frac{dx}{dt} - \gamma v\frac{d\bar{t}}{dt}$$

$$\underbrace{m\frac{d\bar{x}}{d\bar{t}}}_{\bar{p}_x} = \gamma m\underbrace{\left(\frac{dx}{d\bar{t}}\right)}_{\frac{p_x}{1-\frac{v^2}{c^2}}}^{n} - \gamma v m\frac{dt}{dt'}$$

$$F_z = \frac{dp_z}{dt} \rightsquigarrow \bar{F}_z = \frac{d\bar{p}_z}{d\bar{t}} = \frac{F_z}{\gamma\left(1 - \gamma\frac{v}{c^2}u_x\right)}$$

$$\bar{F}_x = \frac{d\bar{p}_x}{d\bar{t}} = \frac{\gamma dp_x - \gamma v}{\qquad\quad ? \qquad\quad}$$

$$x^0 \equiv ct \qquad \beta = \frac{v}{c}$$

$$\bar{t} = \gamma\left(\frac{x^0}{c} - \frac{1}{c}\beta x^1\right) \rightarrow c\bar{t} = \gamma\left(x^0 - \beta x^1\right) \rightarrow \boxed{\bar{x}^0 = \gamma\left(x^0 - \beta x^1\right)}$$

$$\bar{x}^1 = \gamma\left(x^1 - v\frac{x^0}{c}\right) \rightarrow \boxed{\bar{x}^1 = \gamma\left(x^1 - \beta x^0\right)} \qquad \bar{x}^2 = x^2 \quad \bar{x}^3 = x^3$$

$$\begin{pmatrix} \bar{x}^0 \\ \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$